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Mean-field buoyancy

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Abstract. The magnetic field buoyancy is derived in the framework of the mean-field magnetohydrodynamics. The usual linear dependence of the rise velocity on magnetic energy density is found for very weak fields only. The dependence is nonlinear and even non-monotonic for stronger fields. The initial increase of the buoyant velocity with the field strength changes to decrease for strong fields. The decrease is brought about by magnetic tension. The saturation of the mean-field buoyancy weakens substantially the limitations imposed by this effect on cosmic dynamo models. The time of buoyant rise from bottom to top of the solar convection zone is larger than two years for a field of *arbitrary* strength.

Key words: MHD – turbulence – Sun: magnetic fields – stars: magnetic fields

1. Introduction

Magnetic field structures in astrophysical bodies are buoyant (Parker 1975). The buoyancy effect has long been recognized as very important for cosmic magnetic fields dynamics. It largely influences the dynamo models (Schmitt & Schüssler 1989; Moss et al. 1990; Jennings 1991).

However, one can find quite different representations for this effect in the recent publications of dynamo models with buoyancy. The models are usually developed within the mean-field approach. What buoyant rise velocity must be prescribed to the mean field remains very uncertain. The buoyancy effect is well described for the case of magnetic structures (flux tubes) in a quiet non-turbulent atmosphere. The turbulence is probably an unavoidable ingredient of cosmic dynamos however. The magnetic fields in turbulent conducting fluids are known to be intermittent (Meneguzzi et al. 1981; Gilbert 1991; Brandenburg et al. 1991) with some spatially separated flux concentrations. These 'magnetic structures' may be buoyant but their life time is finite. They loose their identity in an eddy turn-over time; what is buoyant then? The question must be certainly reformulated as whether the mean field in the turbulent fluid is subject to buoyancy effect.

It may be argued that presently it is not finally known how long are the life-times of small-scale magnetic structures in stellar convection zones and it is not possible to choose confidently between the two co-existing approaches: the flux-tube dynamo (cf, e.g., Parker 1982; Schüssler 1993) and the mean-field dynamo (Moffatt 1978; Parker 1979; Krause & Rädler 1982). This situation stresses, however, the need for the mean-field buoyancy description which would supply new information allowing to choose between the two alternatives. In addition, the absolute majority of stellar dynamo models belong to the mean-field approach while the buoyancy effect is described for the flux-tubes only.

This paper is aimed at producing the mean-field buoyancy expressions. The gravity and the density fluctuations caused by small-scale magnetic forces are taken into account. The density fluctuations are correlated with that of magnetic field which results in an advection-type term, rot($\mathbf{V}_b \times \mathbf{B}$), in the averaged induction equation. The effective velocity, $\mathbf{V}_b$, is proportional to the fluid compressibility and to the gravity. The drastic difference with the flux-tube buoyancy is the non-monotonous dependence of the effective velocity on magnetic energy density. The usual relation $V_b \sim B^2$ was found for weak fields only. The velocity V_b saturates at a field strength close to the energy equipartition value and decreases with B for still stronger fields. A non-monotonous dependence was found to result from magnetic tension. If only the magnetic pressure part of the complete Lorentz force is kept in the motion equation (the usual practice in the flux-tube buoyancy studies), the relation $V_b \sim B^2$ survives for arbitrary field strengths. Due to the saturation, the mean-field buoyancy is not so dramatic for stellar dynamos as could be expected from the flux-tube buoyancy expressions. The maximal rise velocity for the solar convection zone is about $3\ m\,s^{-1}$ only.

2. Basic equations and assumptions

The turbulence contribute the mean-field induction equation,

$$\partial \mathbf{B}/\partial t = \mathrm{rot}\left(-\eta\,\mathrm{rot}\,\mathbf{B} + \mathscr{E}\right) ,$$

through the mean electromotive force (EMF),

$$\mathscr{E} = \langle \mathbf{u} \times \mathbf{h} \rangle , \tag{2.1}$$

where $\mathbf{B}$ is the mean magnetic field, $\mathbf{u}$ and $\mathbf{h}$ are fluctuating velocity and magnetic field respectively. What we need is the nonlinear expression in B for the mean EMF (2.1) with account for the gravity and compressibility.

We do not wish to overload the treatment by considering the other effects but buoyancy. Several assumptions are made to single-out the conditions in which the buoyancy dominates. The mean velocity (rotation) is assumed zero to exclude the α-effect. The spatial inhomogeneity of averaged fields is neglected. This excludes the diffusive contributions to (2.1) and the turbulent transport effects (diamagnetism).

Nearly all derivations to follow are made within the Second-Order Correlation Approximation (SOCA); the first-order smoothing is another name often used for it. Though this approximation may be (an has been) a subject of criticism, it was never demonstrated to produce unreasonable results and remains the principal tool of the mean-field magnetohydrodynamics. Detailed discussions of the essence and validity limits of the approximation can be easily found in literature (cf., e.g., Moffatt 1978; Krause & Rädler 1980). Hence, we may restrict ourselves to note only that within SOCA the linearized equations for fluctuating fields are used to derive the second-order correlations contributing the mean field equations. In particular, the motion equation can be written as

$$\rho\partial\mathbf{u}/\partial t - \nabla\hat{\sigma} - (\mathbf{B}\cdot\nabla)\mathbf{h}/\mu + \nabla\left(p' + (\mathbf{h}\cdot\mathbf{B})/\mu\right) - \rho'\mathbf{g} = \mathbf{f}\,, \tag{2.2}$$

where ρ and ρ' are the mean and fluctuating densities, $\mathbf{g}$ is the gravity, p' is fluctuating pressure, and $\hat{\sigma}$ is the viscous stress tensor,

$$\sigma_{ij} = \rho\nu\left(\nabla_i u_j + \nabla_j u_i\right) + \rho\xi\delta_{ij}\,\mathrm{div}\,\mathbf{u}$$

(the volume viscosity, ξ, does not contribute the resulting expressions and practically can be omitted). We suppose the buoyancy effect be not sensitive to what is a particular source of the turbulence and prescribe the random body force $\mathbf{f}$ as the turbulence driver.

We assume next the pressure fluctuations to be proportional to that of density:

$$p' = C^2\rho'\,. \tag{2.3}$$

This means that some thermodynamic quantity is not perturbed by the turbulence. E.g., if the heat conductivity is high enough to keep the temperature constant against the turbulent perturbations, we can write

$$C^2 = C_T^2 = \left(\frac{\partial p}{\partial\rho}\right)_T = c_p(\gamma - 1)T/\gamma\,, \tag{2.4a}$$

where $\gamma = c_p/c_v$ is adiabaticity index with c_p and c_v being the specific heats at constant pressure and volume respectively, the ideal gas law,

$$p = (c_p - c_v)\rho T\,,$$

was used. For the opposite case of very low heat conductivity, the adiabatic fluctuations with constant specific entropy, S, can be considered to get

$$C^2 = C_S^2 = \left(\frac{\partial p}{\partial\rho}\right)_S = c_p(\gamma - 1)T\,. \tag{2.4b}$$

It is advantageous in many respects to use Fourier-transformed equations. From (2.2) we find

$$\rho(\nu k^2 - i\omega)\hat{\mathbf{u}} + \rho(\nu + \xi)(\mathbf{k}\cdot\hat{\mathbf{u}})\mathbf{k} - i(\mathbf{B}\cdot\mathbf{k})\hat{\mathbf{h}}/\mu + i\mathbf{k}\left(C^2\hat{\rho}' + (\mathbf{B}\cdot\hat{\mathbf{h}})/\mu\right) - \hat{\rho}'\mathbf{g} = \hat{\mathbf{f}}\,, \tag{2.5}$$

where the hat above letters means Fourier transform, e.g.,

$$\mathbf{u}(\mathbf{r}, t) = \int \exp(i\mathbf{r}\cdot\mathbf{k} - i\omega)\,\hat{\mathbf{u}}(\mathbf{k},\omega)\,d\mathbf{k}\,d\omega\,.$$

We shall need also the equations for fluctuating magnetic field and density. Under the approximations adopted they read

$$\hat{\mathbf{h}}(\mathbf{k},\omega) = \frac{i(\mathbf{B}\cdot\mathbf{k})\hat{\mathbf{u}} - i(\mathbf{k}\cdot\hat{\mathbf{u}})\mathbf{B}}{\eta k^2 - i\omega}\,, \tag{2.6}$$

$$i\omega\hat{\rho}' = i(\mathbf{k}\cdot\hat{\mathbf{u}})\rho\,. \tag{2.7}$$

On using the equation (2.6), the following representation for the mean electromotive force (2.1) can be found,

$$\mathscr{E}_i = i\epsilon_{ijm}\int\int\Big[(\mathbf{B}\cdot\mathbf{k}) < \hat{u}_m(\mathbf{k},\omega)\hat{u}_j(\mathbf{k}',\omega') > - B_m k_p < \hat{u}_p(\mathbf{k},\omega)\hat{u}_j(\mathbf{k}',\omega') > \Big]\frac{d\mathbf{k}\,d\mathbf{k}'\,d\omega\,d\omega'}{\eta k^2 - i\omega}\,. \tag{2.8}$$

The equation of motion should be addressed to define the spectral tensor $< \hat{u}_m\hat{u}_j >$. Substitution of (2.6) and (2.7) into (2.5) yields the closed equation for fluctuating velocity:

$$(\nu k^2 - i\omega)\hat{\mathbf{u}} + (\nu + \xi)(\mathbf{k}\cdot\hat{\mathbf{u}})\mathbf{k} + \frac{1}{\eta k^2 - i\omega}\Big[-(\mathbf{V}\cdot\mathbf{k})(\mathbf{V}\cdot\hat{\mathbf{u}})\mathbf{k} + V^2(\mathbf{k}\cdot\hat{\mathbf{u}})\mathbf{k} + (\mathbf{k}\cdot\mathbf{V})^2\hat{\mathbf{u}} - (\mathbf{k}\cdot\hat{\mathbf{u}})(\mathbf{k}\cdot\mathbf{V})\mathbf{V}\Big] - \frac{(\mathbf{k}\cdot\hat{\mathbf{u}})}{\omega}\mathbf{g} + \frac{iC^2(\mathbf{k}\cdot\hat{\mathbf{u}})}{\omega}\mathbf{k} = \hat{\mathbf{f}}/\rho\,, \tag{2.9}$$

where $\mathbf{V} = \mathbf{B}/(\mu\rho)^{1/2}$ is the Alfven velocity. The Fourier-amplitude, $\hat{\mathbf{u}}$, can be splitted into its solenoidal (incompressible), $\hat{\mathbf{u}}^S$, and potential (compressible), $\hat{\mathbf{u}}^p$, parts:

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}^S + \hat{\mathbf{u}}^p\,,$$
$$\hat{u}_i^S = \pi_{ij}\hat{u}_j,\ \ \hat{u}_i^p = k_i^\circ k_j^\circ\hat{u}_j\,,$$

where $\mathbf{k}^\circ = \mathbf{k}/k$ is the unit vector in the direction of the wave-vector $\mathbf{k}$ and $\pi_{ij}(\mathbf{k}) = \delta_{ij} - k_i^\circ k_j^\circ$ is the projection tensor. Convolutions of the equation (2.9) with π_{ij} and $k_i^\circ k_j^\circ$ yield the equations for the solenoidal and potential constituents of the velocity field,

$$(\nu k^2 - i\omega)\hat{u}_i^S + \frac{1}{\eta k^2 - i\omega}\left[(\mathbf{k}\cdot\mathbf{V})^2\hat{u}_i^S - \pi_{ij}V_j k^2(\mathbf{V}\cdot\hat{\mathbf{u}}^p)\right] - \pi_{ij}g_j\frac{(\mathbf{k}\cdot\hat{\mathbf{u}}^p)}{\omega} = \hat{f}_i^S/\rho \tag{2.10a}$$

$$\left((2\nu+\xi)k^2 - i\omega\right)\hat{u}_i^p + \frac{1}{\eta k^2 - i\omega}\left[\left(k^2V^2 - (\mathbf{k}\cdot\mathbf{V})^2\right)\hat{u}_i^p - k_i(\mathbf{k}\cdot\mathbf{V})(\mathbf{V}\cdot\hat{\mathbf{u}}^S)\right] - \frac{(\mathbf{k}\cdot\mathbf{g})}{\omega}\hat{u}_i^p + \frac{k^2C^2}{\omega}i\hat{u}_i^p = \hat{f}_i^p/\rho . \quad (2.10b)$$

We assume next the fluid compressibility be small, i.e., the 'sound velocity', C, be large as compared to the typical values of the turbulent velocity. In this case, the solution of the system (2.10) can be found by the perturbation method. In the zeroth order in the compressibility the equation (2.10b) reads

$$\frac{k^2C^2}{\omega}i\hat{u}_i^p = 0 , \quad (2.11a)$$

with the obvious result, $\hat{u}^p = 0$. In the same approximation the equation (2.10a) produces the well-known (cf. Krause & Rädler 1980) relation for incompressible turbulence affected by the mean magnetic field:

$$\hat{\mathbf{u}}^S = \hat{\mathbf{u}}^{S(0)}/N , \quad (2.11b)$$

where

$$\hat{\mathbf{u}}^{S(0)} = \frac{\hat{\mathbf{f}}^S}{(\nu k^2 - i\omega)\rho} , \quad N = 1 + \frac{(\mathbf{k}\cdot\mathbf{V})^2}{(\eta k^2 - i\omega)(\nu k^2 - i\omega)} .$$

$\hat{\mathbf{u}}^{S(0)}$ can be identified with the velocity amplitude of the so-called 'original turbulence' (Rüdiger 1989), i.e., with the velocity field which would take place under real sources of turbulence but if the mean magnetic field were absent.

Obviously, finite compressibility must be included to account for the buoyancy effect. We have to do the next step in the perturbation procedure. This yields

$$\hat{u}_i^p = \hat{u}_i^{p(0)} + k_i\frac{\omega(\mathbf{k}\cdot\mathbf{V})(\mathbf{V}\cdot\hat{\mathbf{u}}^{S(0)})}{ik^2C^2(\eta k^2 - i\omega)N} , \quad (2.12a)$$

$$\hat{u}_i^S = \frac{\hat{u}_i^{S(0)}}{N} + \frac{\pi_{ij}(\mathbf{k}\cdot\hat{\mathbf{u}}^{p(0)})}{N(\nu k^2 - i\omega)}\left(\frac{(\mathbf{k}\cdot\mathbf{V})}{\eta k^2 - i\omega}V_j + \frac{g_j}{\omega}\right) + \frac{\pi_{ij}(\mathbf{k}\cdot\mathbf{V})(\mathbf{V}\cdot\hat{\mathbf{u}}^{S(0)})}{iC^2N^2(\nu k^2 - i\omega)(\eta k^2 - i\omega)}\left(\frac{\omega(\mathbf{k}\cdot\mathbf{V})}{\eta k^2 - i\omega}V_j + g_j\right) , \quad (2.12b)$$

where

$$\hat{u}_i^{p(0)} = \frac{\omega}{i\rho k^2C^2}\hat{f}_i^p \quad (2.13)$$

is the potential part of the velocity amplitude for the original turbulence.

It remains to define the original turbulence. The turbulence is assumed statistically steady and homogeneous,

$$< \hat{u}_i^{(0)}(\mathbf{k},\omega)\,\hat{u}_j^{(0)}(\mathbf{k}',\omega') > = Q_{ij}(\mathbf{k},\omega)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') ,$$

and spatially-isotropic,

$$Q_{ij}(\mathbf{k},\omega) = \left(E(k,\omega)\pi_{ij}(\mathbf{k}) + 2E^p(k,\omega)k_i^\circ k_j^\circ\right) / (16\pi k^2) . \quad (2.14)$$

E and E^p are the spectra for incompressible and compressible parts of the turbulence, respectively, which both contribute to the total turbulence intensity:

$$< (u^{(0)})^2 > = \int_0^\infty (E+E^p)\,dk\,d\omega ,$$

$$< (u^{S(0)})^2 > = \int_0^\infty E\,dk\,d\omega , \quad < (u^{p(0)})^2 > = \int_0^\infty E^p\,dk\,d\omega .$$

It may be seen from (2.13) that the spectrum E^p is of the second-order in the fluid compressibility. We include it in (2.14) for completeness only. It does not contribute to the final results.

It can be easily shown that the 'incompressible' solution (2.11) is such that the mean electromotive force (2.8) equals zero. Thus, we really have no mean-field transportation effect without density fluctuations.

3. The buoyant rise velocity

3.1. The general expressions and their interpretation

The non-trivial representation for the mean EMF (2.1) can be found by substitution of the 'compressible' solution (2.12) and (2.14) into (2.8). After some algebra we derive the advection-type expression,

$$\mathcal{E} = \mathbf{V}_b \times \mathbf{B} , \quad (3.1)$$

with the following representation for the effective velocity:

$$\mathbf{V}_b = -\mathbf{g}\frac{1}{C^2}\int_0^\infty\int_0^\infty \frac{\eta k^2 E(k,\omega)}{\eta^2k^4+\omega^2}\beta^2 K(k,\omega,\beta)\,dk\,d\omega , \quad (3.2)$$

where

$$\beta = \frac{kV}{(\nu^2k^4+\omega^2)^{1/4}(\eta^2k^2+\omega^2)^{1/4}} \quad (3.3)$$

is the normalized field parameter. As expected, the velocity is proportional to both the gravity, $\mathbf{g}$, and the fluid compressibility, C^{-2}. The magnetic field dependence enters, however, in a rather complicated way through the kernel function

$$K = \frac{1}{16\beta^4}\left\{\frac{\beta^2-3}{2\beta\cos(\phi/2)}\left[\arctan\left(\frac{\beta+\sin(\phi/2)}{\cos(\phi/2)}\right) + \arctan\left(\frac{\beta-\sin(\phi/2)}{\cos(\phi/2)}\right)\right] - \frac{(\beta^2+3)}{4\beta\sin(\phi/2)}\ln\left(\frac{\beta^2-2\beta\sin(\phi/2)+1}{\beta^2+2\sin(\phi/2)+1}\right)\right\} , \quad (3.4)$$

where we introduced the 'angle' ϕ defined by the relation

$$\cos(\phi) = \frac{\nu\eta k^4 - \omega^2}{(\nu^2k^4+\omega^2)^{1/2}(\eta^2k^4+\omega^2)^{1/2}} .$$

The usual result is a buoyant rise velocity proportional to the second power of the magnetic field strength. With equation (3.2) this is the case for weak fields only. When β (3.3) is small for those wave numbers and frequencies which give the main contribution to the spectral integral in (3.2), we can put $\beta \to 0$ in (3.4) to find

$$K = \cos\phi/15 .$$

Then, for the weak field case we find

$$\mathbf{V}_b = -\frac{\mathbf{g}B^2}{15C^2\mu\rho}\int_0^\infty\int_0^\infty \frac{\eta k^4(\nu\eta k^4-\omega^2)E(k,\omega)}{(\eta^2k^4+\omega^2)^2(\nu^2k^4+\omega^2)}dk\,d\omega. \quad (3.5)$$

For the opposite case of large β we can keep in (3.4) only the lowest-order term in the small parameter β^{-1} to find,

$$K = \frac{\pi}{32\beta^3\cos(\phi/2)} .$$

Rather unexpectedly, the velocity (3.2) decreases with B in this strong magnetic field case:

$$\mathbf{V}_b = -\mathbf{g}\frac{\pi(2\mu\rho)^{1/2}}{32BC^2}\int_0^\infty\int_0^\infty \frac{\eta k E}{\eta^2k^4+\omega^2} \left(\frac{(\nu^2k^4+\omega^2)(\eta^2k^4+\omega^2)}{(\nu^2k^4+\omega^2)^{1/2}(\eta^2k^4+\omega^2)^{1/2}+\nu\eta k^4-\omega^2}\right)^{1/2} dk\,d\omega. \quad (3.6)$$

This result is in drastic contrast with the usual relation $V_b \sim B^2$. The discrepancy is certainly rooted in the difference of our approach from that producing the B^2-law. The key point is the inclusion of the magnetic tension forces which are usually ignored in buoyancy studies. If the above derivations are repeated by taking only the magnetic pressure part of the Lorentz force into account, the B^2-expression (3.5) results for arbitrary field strength. Therefore, the non-monotonous dependence of V_b on B results from magnetic tension. (We use the term 'tension' in the sense that the complete Lorentz force = magnetic pressure + magnetic tension.) It is worth noting that if the pressure is neglected but tension included then no mean-field transport results, $V_b = 0$. Hence, the mean-field buoyancy is produced solely by magnetic pressure. Only the tension modifies the effect but the modification is very strong.

The following qualitative explanation of the result may be suggested. Turbulent wiggling of the mean-field lines of force produces random magnetic inhomogeneities. The magnetic pressure is higher than the average value in the regions of concentrated flux and lower than the average in the flux depletion regions. The resulting density fluctuations and gravity cause the flux concentrations to rise somewhat and the flux depletions to sink somewhat during the flux irregularities life-time. The rising flux is stronger. On average, the flux is transported upwards producing the mean-field buoyancy. For a given spectrum of magnetic irregularities, only magnetic pressure is efficient to produce the density inhomogeneities and the resulting buoyancy effect. For the irregularities formation, however, magnetic tension is important. If the mean-field is very strong, the irregularities are difficult to produce against the magnetic tension and the buoyancy is suppressed.

3.2. The mixing-length approximation

The above expressions for the buoyant velocity should be further adapted for application purposes because they include the spectral function E and other parameters purely known for real objects. The mixing length approximation is convenient for application purposes. We arrive at one of the variants of this approximation if replace d/dt in the starting equations for fluctuating fields by τ^{-1} (cf. Durney & Spruit 1979) and consider a single-scaled spectrum $E \sim \delta(k - l^{-1})$; τ and l being the convective turn-over time and mixing length respectively. It is not necessary, however, to repeat all the derivations with these new equations to find the mixing-length representation of the above results. We may found them formally (Kichatinov 1991) by substituting the following expressions,

$$E(k,\omega) = 2 < (u^{(0)})^2 > \delta(k-l^{-1})\delta(\omega),\, \nu k^2 = \eta k^2 = \tau^{-1}, \quad (3.7)$$

into the above equations. Thus obtained mixing-length representation of the velocity (3.2) reads

$$\mathbf{V}_b = -\mathbf{g} < (u^{(0)})^2 > \tau\beta^2 K(\beta)/C^2 , \quad (3.8)$$

where K and β are the mixing-length analogues of (3.3) and (3.4):

$$K(\beta) = \frac{1}{16\beta^4}\left(1+\frac{2}{1+\beta^2}+\frac{\beta^2-3}{\beta}\arctan(\beta)\right) , \quad (3.9)$$

and $\beta = V\tau/l$ is the field strength normalized to the energy equipartition value. The function (3.9) tends to a constant,

$$K = 1/15 , \quad (3.10)$$

when β is small and decreases with β as

$$K = \pi/(32\beta^3) , \quad (3.11)$$

in the strong field limit ($\beta \gg 1$) to reproduce the simplified versions of the above expressions (3.5) and (3.6) in the mixing-length approximation.

4. In a stellar convection zone

4.1. The 'original turbulence'

The velocity $u^{(0)}$ corresponds to the original turbulence. By definition, this is the turbulence which would take place under the actual source of the turbulence but if the mean magnetic field where absent. In the stellar convection zones the source is the superadiabaticity of stratification. The known mixing-length relation,

$$< (u^{(0)})^2 >= -\frac{l^2 g}{4T}\frac{\partial\Delta T}{\partial r} , \quad (4.1)$$

can be applied to estimate the original turbulence intensity. A complication comes however from the mean magnetic field influence on convective heat transport and the resulting superadiabatic temperature gradient, $\nabla\Delta T$. For this reason the original turbulence still depends on the magnetic field. The aim of this subsection is to define the dependence explicitly.

We start from the mixing-length representation for the convective heat flux introduced by Wasiutynski (1946),

$$\mathbf{F} = -\rho\tau c_p < \mathbf{u} u_i > \nabla_i \Delta T \, . \tag{4.2}$$

The velocity correlation tensor is needed to derive the (tensorial) heat conductivity. The small fluid compressibility can be neglected now and the equations (2.11), (2.14) and (3.7) can be used to find

$$< u_i u_j > = < (u^{(0)})^2 > \left(\frac{1}{3} \Psi(\beta) \delta_{ij} + \Psi_1(\beta) \frac{B_i B_j}{B^2} \right) , \tag{4.3}$$

where the factor 1/3 is introduced for convenience. Only the function

$$\Psi(\beta) = \frac{3}{8\beta^2} \left(\frac{\beta^2 - 1}{\beta^2 + 1} + \frac{\beta^2 + 1}{\beta} \arctan(\beta) \right) , \tag{4.4}$$

of the two contributing fuctions of β will be needed. If the dominating component of the mean magnetic field is the toroidal one, the equations (4.1), (4.2) and (4.3) give

$$\begin{aligned} F_r &= -\rho c_p \tau < (u^{(0)})^2 > \frac{\Psi}{3} \frac{\partial \Delta T}{\partial r} \\ &= \Psi(\beta) \frac{\rho c_p \tau g l^2}{12T} \left(\frac{\partial \Delta T}{\partial r} \right)^2 . \end{aligned} \tag{4.5}$$

It seems natural to assume that the mean magnetic field does not suppress the convective heat flux. Otherwise the stellar luminosity would be modulated with the magnetic cycle period. The variations for the Sun, if any, are extremely small (Stix 1989). The physical reason for this assumption is the nonlinear nature of the convective heat transport: suppression of the heat flux would result in an increased superadiabatic temperature gradient which strongly opposes suppression by amplifying the convective mixing. Our final assumption is the independence of the convective turn-over time τ on the magnetic field. Under these assumptions we find

$$\left(\frac{\partial \Delta T}{\partial r} \right)^2 = \left(\frac{\partial \Delta T}{\partial r} \right)_0^2 \Psi^{-1}(\beta) , \tag{4.6}$$

where the subscript '0' means the value for zero magnetic field. Combining (4.1) and (4.6) we find

$$< (u^{(0)})^2 > = u'^2 / \Psi^{1/2}(\beta) , \tag{4.7}$$

where u' is the rms velocity for the nonmagnetic case which can be taken from the (nonmagnetic) stellar convection zone models (cf., e.g., Baker & Temesvary 1966). Eq. (4.7) describes the original turbulence dependence on the magnetic field. The dependence is very weak. For $\beta = 1$, $\Psi = 3\pi/16$, i.e, $u^{(0)}$ and u' differ less than by a factor of 3/4. The magnetic modification is larger for $\beta \gg 1$ but a stellar dynamo can hardly produce the very strong field regime. Nevertheless, we shall take the weak magnetic modification (4.7) of the original turbulence into account when estimating the buoyant velocities.

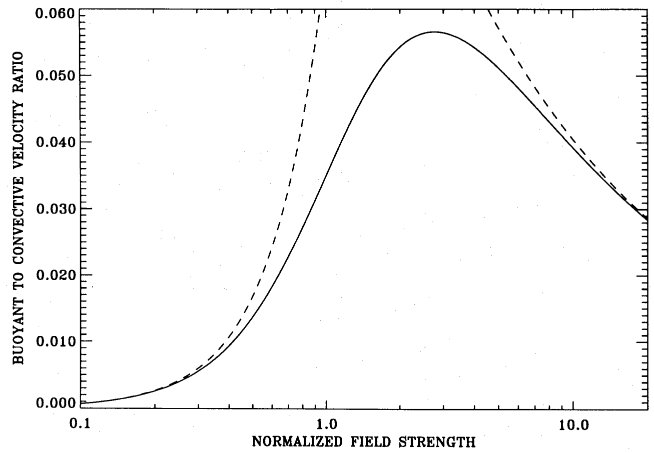

Fig. 1. The buoyant-to-convective velocities ratio as a function of the normalized field $\beta = B/[(\mu\rho)^{1/2}u']$. The broken line shows the asymptotics (4.13) and (4.14) for the weak and strong field cases

4.2. *Estimating the rise velocities*

The deviation of the stellar convection zones stratification from adiabaticity is very small except for the near-top layer with partial ionization. E.g., the relative value of this deviation for the Sun is about 10^{-4} (Stix 1989). We find from (2.4) for this case

$$g/C^2 = (\kappa H)^{-1} , \tag{4.8}$$

where $H = (\gamma - 1)c_p T/(\gamma g)$ is the pressure scale height and the parameter κ accounts for the thermodynamic properties of the turbulence: $\kappa = \gamma$ for adiabatic and $\kappa = 1$ for isothermal fluctuations. Substitution of (4.7) and (4.8) into (3.8) gives the following expression for the absolute value of the rise velocity,

$$V_b = \frac{l V^2}{15 \kappa H u'} Q(\beta) , \tag{4.9}$$

where the usual relation $\tau = l/u'$ was used, the function $Q(\beta)$ is expressed through the functions K and Ψ defined by (3.9) and (4.4),

$$Q(\beta) = 15 K(\beta) / \Psi^{1/2}(\beta) . \tag{4.10}$$

The factor 15 was introduced to normalize Q to unity at the origin, $Q(0) = 1$. With this choice, Q can be understood as the quenching function of the buoyancy effect by magnetic tension.

In a more consistent account for the thermodynamic properties of the turbulence one could use the energy equation instead of the simple relation (2.3). However, we can find from (4.9) that the rise velocity changes by a factor of 5/3 only between the opposite limits of very high ($\kappa = 1$) and very low ($\kappa = 5/3$) temperature conductivities. It would be rather strange if somewhere between this limits the relation (4.9) was strongly violated. Note that the velocity (4.9) keep the isothermal fluctuations is slightly amplified as compared to the adiabatic fluctuations.

The equation (4.9) is similar to the flux-tubes rise velocity expression (Parker 1978),

$$V_{tube} = c\frac{R^2V^2}{lHu'} . \tag{4.11}$$

where R is the tube radius and c is a constant of order unity. Note that in our case the magnetic structure 'radius' is the turbulence scale l. There are, however, two significant differences. First, the mean-field buoyancy is a statistical effect. Probably for this reason the velocity (4.9) is about an order of magnitude smaller for the weak field case. Second, the expression (4.9) includes the buoyancy quenching and the difference with (4.11) is amplified when the field is not weak.

The mixing-length is usually supposed to be roughly equal to the pressure scale height. In this case the ratio of the buoyant to convective velocity can be found from (4.9) with an accuracy within a factor close to unity to equal

$$V_b/u' = \beta^2 Q(\beta)/15 . \tag{4.12}$$

The ratio is very small for both extremes of weak,

$$V_b/u' = \beta^2/15 \quad \text{for } \beta \ll 1 , \tag{4.13}$$

and strong fields,

$$\frac{V_b}{u'} = \frac{1}{8}\left(\frac{\pi}{3\beta}\right)^{1/2} \quad \text{for } \beta \gg 1 . \tag{4.14}$$

The ratio (4.12) is shown in Fig.1 as a function of β. The buoyant velocity cannot be larger than about 6 percent of the convective velocity. Moreover, it is rather improbable that a stellar dynamo can reach the $\beta > 1$ region where the maximal value is placed. E.g., if we adopt $B \simeq 2\,kG$ for the Sun, the parameter β is slightly smaller than unity over the entire convection zone except for very thin layers at the top and bottom (cf., e.g., the SCZ model by Spruit 1974). $V_b/u' \simeq 0.02$ is probably a plausible value for the Sun. This means that with $u' \simeq 50\ m\ s^{-1}$ for the large-scale solar convection (Stix 1989) we find the rise velocity of about $1\ m\ s^{-1}$ only. In any case the velocity cannot be larger than about $3\ m\ s^{-1}$. Even with this latter value the time of rise from the bottom to top of the SCZ is about 2 yr; 6 yr seems to be a more reasonable evaluation.

The estimations indicate that the mean-field buoyancy may considerably influence the magnetic fields on long time scales but in no case the buoyancy is so deeply disastrous for the stellar dynamos as it is often claimed.

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