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Toward the theory of magnetospheric magnetosonic eigenoscillations: simple theoretical models

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Abstract

The structure of magnetosonic eigenmodes was investigated analytically within the framework of simple two- and three-dimensionally inhomogeneous models. Asymptotic (in the limit $m \gg 1$, where m is azimuthal wave number) expressions were obtained for the eigenfrequency spectrum of magnetosonic oscillations in the cavity resonator under the plasmapause, as well as for outer magnetospheric cavity modes, the outer boundary for which is represented by the magnetopause. These oscillations were found to be damped ones. Furthermore, the damping of the eigenoscillations in the cavity under the plasmapause is caused by the escape of some of their energy through the potential barrier to the outer magnetosphere. The damping of the outer magnetospheric cavity modes is due to the fact that the magnetopause represents a partially reflecting boundary, through which a part of the oscillation energy escapes to the solar wind. In terms of an axisymmetric model of the magnetosphere, asymptotic (when $m \gg 1$) expressions were obtained for damping decrements of the eigenmodes. The spatial structure of magnetosonic eigenmodes was investigated in a simple three-dimensionally inhomogeneous model of the magnetosphere. The asymptotic expression was obtained for the spectrum of their eigenfrequencies.

1. Introduction

It is known that a homogeneous magnetized plasma supports three independent branches of hydromagnetic oscillations: Alfven waves and fast and slow magnetosonic waves. Oscillations of the same types exist also in magnetospheric inhomogeneous plasma. The inhomogeneity has a determining effect on the properties of these oscillations, thus adding complexity to their theoretical study. Even in one-dimensionally inhomogeneous models of the medium (plane plasma layer models), the currently available theory [*Radoski*, 1974; *Southwood*, 1974; *Chen and Hasegawa*, 1974] encounters considerable difficulty.

This is especially true in regard to magnetosonic oscillations. A qualitative analysis of the transverse structure of magnetosonic eigenocillations in the box model was made by *Lee and Kim* [1999]. They drew the analogy between the solution of the Schrodinger equation for quantum oscillators and the solution of the wave equation for magnetosonic oscillations of the magnetosphere. For two-dimensional, not to mention three-dimensional, models the theory is still far from full grown. In this paper we undertake an investigation for an axisymmetric (two-dimensionally inhomogeneous) model of the magnetosphere and perform some calculations for a simple three-dimensionally inhomogeneous model. The plasma will be assumed as sufficiently cold, $\beta \ll 1$, which is adequately justified for the inner part of the magnetosphere.

The condition $\beta \ll 1$ permits us to neglect the role of slow magnetosonic waves, as it represents in this case very slow waves which cannot be global oscillations of the magnetosphere. The remaining two types of geomagnetic oscillations, Alfven (shear) and fast magnetosonic waves, are coupled oscillations. For monochromatic waves (waves with a given frequency ω) this coupling is important only near resonance magnetic shells where the magnetosonic wave frequency coincides with the Alfven eigenfrequency of the shell. The theory of Alfven (shear) oscillations of the axisymmetric magnetosphere is presently on a sufficiently firm, detailed footing [*Krylov et al.*, 1981; *Leonovich and Mazur*, 1989; *Chen and Cowley*, 1989; *Wright*, 1992]. The properties of these oscillations depend substantially on the value of azimuthal wave number m.

When $m \sim 1$, the source of the Alfven (shear) wave is provided by fast magnetosonic oscillations that penetrate the magnetosphere from its boundary or from the solar wind. These waves have often been shown to be field line resonance. The resonance surface, near which the Alfven wave is excited, lies in the opaque region of magnetosonic wave at a distance of about the wavelength from the boundary of the opaque region. The amplitude of magnetosonic oscillations decreases exponentially with distance from the boundary into the opaque region. An exponential decrease in the amplitude of magnetosonic oscillations deep into the opaque region is relatively small, which permits the oscillations to play the role of a source for the Alfven (shear) wave. It was shown by *Leonovich and Mazur* [1989] that the back influence of the excited Alfven waves on the structure of magnetosonic oscillations is unimportant, even near the resonance surface. The main reason for this is that the resonance surface lies in the opaque region of magnetosonic wave. This influence will be analyzed in greater detail in our next paper.

When $m \gg 1$, the resonance surface lies deep in the interior of the opaque region, to which magnetosonic oscillations penetrate almost not at all. Therefore the source of Alfven (shear) waves in this region must have its origin inside the magnetosphere, such as external currents in the ionosphere. The theory of such Alfven oscillations was developed in detail by *Leonovich and Mazur* [1993]. Success of the theory of Alfven oscillations in the axisymmetric model of the magnetosphere derives largely from the fact that they are "tied" to geomagnetic field lines. This makes it possible to separate the problem into the study of the longitudinal structure of the wave and the study of the transverse (with respect to the geomagnetic field) structure. The two-dimensional problem of investigating this structure is reducible to two sequential one-dimensional problems, each of which allows for a relatively simple solution.

A distinctly different situation occurs in the case of magnetosonic oscillations. It is in no way tied to field lines, and the problem of investigating it in the axisymmetric model is essentially a two-dimensional problem alone. Such two-dimensional problems are known to have no general analytical solution. In this connection the structure of magnetosonic oscillations in such models of the magnetosphere has been investigated to date principally by numerical simulation methods [*Allan et al.*, 1986; *Lee and Lysak*, 1989; *Lee and Lysak*, 1991; *Lee*, 1996]. It is not our intent here to make an in-depth analysis of the advantages and disadvantages of such an approach. The only point that should be mentioned is that in addition to such numerical models it is always worthwhile to have a simplified analytical model enabling a qualitative study to be made of the main features of the oscillations under consideration, in a medium with the same dimensions.

In this paper we take just this approach to studying the magnetosonic eigenocillations in the axisymmetric and the three-dimensionally inhomogeneous models of the magnetosphere. Specifically, we investigate some limiting cases where the oscillations are concentrated within narrow regions of the magnetosphere. These limiting cases enable one to make an accurate (as well as sufficiently simple) investigation. At the same time, at the limit of their applicability these solutions provide a qualitative idea of the general case where the region of localization of the oscillations is comparable to the size of the magnetosphere.

2. Regions of Localization of Magnetosonic Eigenocillations

We shall use a usual cylindrical coordinate system (r, z, φ), where r is the distance from the axis of symmetry, z is a coordinate along this axis, and φ is the azimuthal angle. The axisymmetric model of the magnetosphere assumes that equilibrium parameters are independent of the angle φ. In the limit of ideal MHD the disturbed electric field of the wave $\mathbf{E}$ is normal to the geomagnetic field; that is, in our selected coordinate system it has only two components, E_r and E_φ. The two-component vector $\mathbf{E}_\perp$ may be represented as the superposition of the vortical and potential components,

$$\mathbf{E}_\perp = -\nabla_\perp \Phi + [\nabla_\perp \times \mathbf{b}\Psi],$$

where $\mathbf{b}$ is a unit vector in the direction along the geomagnetic field and Φ and Ψ are functions of coordinates. As is known [*Klimushkin*, 1994; *Fedorov*, 1998], in a homogeneous cold plasma the potential Φ describes the Alfven (shear) wave, and Ψ represents magnetosonic wave (i.e., $\Psi = 0$ in Alfven oscillations, and $\Phi = 0$ in magnetosonic oscillations). In an inhomogeneous plasma these oscillations are no longer independent ones, and their separation into the Alfven and magnetosonic waves becomes something of a matter of convention.

As pointed out in the section 1, in studying magnetosonic oscillations we will neglect its connection with the Alfven (shear) wave. In this case the equation for monochromatic magnetosonic oscillations $\Psi \sim e^{-\mathrm{i}\omega \mathrm{t}}$ takes the form

$$\Delta\Psi + \frac{\omega^2}{A^2}\Psi = 0. \qquad (1)$$

Here $\Delta \equiv r^{-1}\nabla_r r \nabla_r + r^{-2}\nabla_\varphi^2 + \nabla_z^2$ - Laplacian, and $A = B_0/\sqrt{4\pi\rho}$ is the Alfven velocity that is a function of coordinates r and z. In (1) some simplifications are introduced, based on the assumption that the function Ψ varies in coordinates r, z significantly faster when compared to A. This assumption is based on the fact that the analytical solutions of (1) will be considered in those limiting cases in which these solutions are localized in coordinates r and z in the narrow vicinities of some magnetosphere regions. At the limit of their applicability these solutions describe qualitatively correctly the oscillations with a scale of the order of the size of the magnetosphere (equation (1) is applicable with the same accuracy).

Using the axial symmetry, we now proceed to examining the oscillations that are azimuthal harmonics $\Psi \sim e^{-im\varphi}$. For them, (1) becomes

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Psi}{\partial r} + \frac{\partial^2 \Psi}{\partial z^2} + \left(\frac{\omega^2}{A^2} - \frac{m^2}{r^2}\right)\Psi = 0. \qquad (2)$$

Equation (2) coincides with the two-dimensional Schrodinger equation,

$$\Delta\Psi + (E - U) = 0,$$

where Δ is the two-dimensional Laplacian operator in coordinates r and z, E is the energy level, and $U = U(r, z)$ is potential energy. In the case of interest,

$$E = 0, \qquad U = \frac{m^2}{r^2} - \frac{\omega^2}{A^2}. \qquad (3)$$

In other words, the problem of solving (2) coincides with that of finding the quantum eigenstate with the energy $E = 0$ in the potential U. The role of the eigenvalue is played by the frequency ω, which must be chosen such that the potential energy in (3) would allow for a state with a zero energy level $E = 0$.

To explore the possibilities available to us, first we consider the equatorial profile of potential energy. Let $z = 0$ hold on the equator. The potential energy U may be represented as

$$U = \omega^2 u, \qquad u = \frac{\mu^2}{r^2} - \frac{1}{A^2}, \qquad \mu = \frac{m}{\omega}.$$

The function u is more conveniently used in the analysis, because it depends only on one parameter μ.

A typical form of the equatorial profile of the Alfven velocity of moderately disturbed dayside magnetosphere $A(r,0)$ is presented in Fig.1. [see *Horwitz et al.*, 1986; *Comfort*, 1986]. The distinctive features of this plot are the following: a maximum $A_0 = (1.3) \times 10^3$ km s^{-1} at the point $r_0 \approx 1.3 R_E$; a minimum $A_p \approx 400$ km s^{-1} on the inner edge of the plasmapause when $r_p = 3.1\, R_E$ and a maximum $A_{p1} \approx 10^3$ km s^{-1} on its outer edge; and an abrupt change from $A_m \approx 500$ km s^{-1} to $A_{m1} \approx 50$ km s^{-1} at the magnetopause when $r_m \approx 10\, R_E$. In the magnetosphere these parameters can vary depending on the level of geomagnetic disturbance: $A_0 = (1-4) \times 10^3$ km s^{-1}, $A_p = (100-500)$ km s^{-1}, $r_p = (3-6)\, R_E$, and $r_m = (7-12)\, R_E$.

Depending on the value of the parameter μ, there are three different possibilities for the behavior of the function

$$u(r,0) = \frac{\mu^2}{r^2} - \frac{1}{A^2(r,0)}.$$

Case a occurs when $\mu < r_{p1}/A_{p1}$, case b is realized when $r_{p1}/A_{p1} < \mu < r_p/A_p$, and case c occurs when $\mu > r_p/A_p$.

It can be seen that in case b there is a potential well in the neighborhood of a minimum Alfven velocity below the plasmapause; this well is bounded on both sides by turning points, at which $u = 0$. The existence of an eigenmode is possible in the area of this well. This is a well-known waveguide for magnetosonic waves under the plasmapause [*Gul'elmi*, 1970; *Gul'elmi*, 1972]. Furthermore, the three cases involve the possible existence of an eigenmode bounded on the outer side by a sharp boundary on the magnetopause and on the inner side by the usual turning point, at which $u = 0$. In the literature available to date, such oscillations have come to be known as global modes [*Kivelson and Southwood*, 1986; *Southwood and Kivelson*, 1986]. Of course, it is conceivable that the eigenmode can also be confined in the magnetosphere because of the presence of a traveling plasma in the solar wind [*McKenzie*, 1970]. We will explore this possibility in our subsequent work; in this paper we limit ourselves to a simple model, in which we neglect the plasma motion.

Let us now recall that the function $u = u(r,z)$ is a two-dimensional one and that its behavior needs to be investigated also in coordinate z. Figure 2 presents a typical form of the level lines u for case b. It can be seen that the potential well under the plasmapause is bounded along field lines as well. This refers also to the region of localization of the eigenmode at the outer magnetosphere bounded by an abrupt jump at the magnetopause.

In both of the above cases the eigenmodes are damped ones because of their possibility of flowing out of the region of localization. This possibility for the wave localized under the plasmapause is associated with its partial penetration through the potential barrier, and this possibility for the wave bounded by the magnetopause is associated with the fact that only partial reflection occurs from the sharp boundary. Let us now consider in detail both possibilities of existence of the eigenmodes.

3. The Eigenmode Under the Plasmapause

We will seek an accurate analytical solution for the mode localized near the very bottom of the potential well; the model of the Alfven velocity $A(r,z)$ near this bottom would suffice to do this. We use the coordinate $x = r - r_p$, where r_p is a coordinate of the plasmapause. In the region of our interest we use the following model of the Alfven velocity

$$A = \begin{cases} A_p[1 - (x/2a_p) + (z^2/2l_p^2)] & x < 0, \\ A_{p1}[1 - (x/2a_{p1}) + (z^2/2l_{p1}^2)] & x > 0, \end{cases} \tag{4}$$

where $A_{p1} > A_p$. Equation (4) reflects qualitatively correctly the familiar "knee" on the profile $A(x,0)$. The parameters a_p, a_{p1}, l_p, l_{p1} represent the characteristic scales in coordinates x and z, respectively. They are order-of-magnitude coincident with the radius of the plasmapause r_p.

On expanding to terms linear in x and quadratic in z, for the potential energy we get

$$U = \begin{cases} (m/r_p)^2 - (\omega/A_p)^2[1 + (x/a_p) - (z/l_p)^2] \\ \qquad x < 0, \\ (m/r_p)^2 - (\omega/A_{p1})^2[1 + (x/a_{p1}) - (z/l_{p1})^2] \\ \qquad x > 0, \end{cases} \tag{5}$$

The equatorial profile of this function is presented in Figure 3.

The eigenmode can be localized near the bottom of the well if $\omega^2 = m^2 A_p^2/r_p^2$. In the main order of the approximation used here, it will also be assumed that the potential barrier to the right of the potential well is large enough for $U = \infty$ to hold when $x > 0$. We have

$$U = \begin{cases} -\varepsilon - kx + gz^2 & x < 0, \\ \infty & x > 0. \end{cases}$$

The following designations are used here:

$$\varepsilon = \frac{\omega^2}{A_p^2} - \frac{m^2}{r_p^2}, \qquad k = \frac{m^2}{a_p r_p^2}, \qquad g = \frac{m^2}{l_p^2 r_p^2}. \quad (6)$$

In the order of magnitude $k \sim m^2/r_p^3$, $\quad g \sim m^2/r_p^4$. Equation (2) when $x < 0$ has the form

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial z^2} + (\varepsilon + kx - gz^2)\Psi = 0, \qquad (7)$$

and the boundary conditions for (7) are

$$\Psi(x = 0, z) = \Psi(x \to -\infty, z) = \Psi(x, z \to \pm\infty) = 0. \quad (8)$$

The first condition is determined by the fact that under our assumption the potential barrier is infinitely high, and the other two conditions arise from the requirement that the amplitude of the eigenmode should decrease with the distance from the place of its localization. Note that in the problem (7), (8) the parameter ε plays the role of an eigenvalue. The frequency ω is determined through this parameter in accordance with the first relation in (6).

Equation (7), together with the boundary conditions (8), represents problems with separate variables. This means that the solution is represented as the production of a function dependent only on x by a function dependent only on z; these functions are briefly described below. Let $\psi = \psi_-(\xi)$ be the real solution of the equation

$$\psi'' + \xi\psi = 0, \qquad (9)$$

satisfying the boundary condition $\psi_-(\xi)|_{\xi \to -\infty} = 0$. This solution (up to an arbitrary factor) is the Airy function $\psi_-(\xi) = Ai(\xi)$. When $\xi > 0$, the function $\psi_-(\xi)$ is an oscillating one and has an infinite number of zeroes ξ_j $(j = 0, 1, \dots)$. Let us introduce the function

$$\psi_j(\xi) = \psi_-(\xi + \xi_j). \qquad (10)$$

Equation (10) satisfies the equation

$$\psi_j'' + (\xi + \xi_j)\psi_j = 0$$

and the boundary condition $\psi_j(0) = 0$, $\psi_j(\xi)|_{\xi \to -\infty} = 0$. This function has j zeroes on the semi-axis $(-\infty, 0)$.

The second function $\varphi(\eta)$, which will be required in our further discussion, is the solution of the following eigenvalue problem:

$$\varphi'' + (\lambda - \eta^2)\varphi = 0, \qquad \varphi(\eta)|_{\eta \to \pm\infty} = 0.$$

Here λ is the desired eigenvalue. As is known [*Landau and Lifshitz*, 1963], this function has the following set of eigensolutions

$$\begin{aligned} \lambda &= \lambda_n = 2n + 1, \\ \varphi &= \varphi_n(\eta) \equiv e^{-\eta^2/2} H_n(\eta), \end{aligned}$$

$n = 0, 1, \dots,$ where $H_n(\eta)$ are Hermite polynomials. The function $\varphi_n(\eta)$ has n zeroes on the interval $(-\infty, \infty)$.

The solution of the problem (7),(8) is representable as

$$\Psi = C\psi_j\left(\frac{x}{\alpha}\right)\varphi_n\left(\frac{z}{\beta}\right). \qquad (11)$$

The boundary conditions (8) are satisfied automatically. Equation (7) is also satisfied when

$$\alpha^{-1} = k^{1/3} = m^{2/3}/a_p^{1/3} r_p^{2/3}, \qquad (12)$$

$$\beta^{-1} = g^{1/4} = m^{1/2}/l_p^{1/2} r_p^{1/2}, \qquad (13)$$

and at the eigenvalues

$$\varepsilon = \frac{\xi_j}{\alpha^2} + \frac{\lambda_n}{\beta^2}, \qquad (14)$$

The parameters α and β play the role of characteristic scales of the eigensolution in coordinates x and z, respectively. In the order of magnitude $\alpha \sim r_p/m^{2/3}$, $\beta \sim r_p/m^{1/2}$. So that our assumption (that the characteristic scale of variation of the solution is much smaller than that of magnetospheric parameters) will hold, it is necessary that the condition $m \gg 1$ be satisfied.

In this case the relations (11)-(13) describe the magnetosonic eigenwave that represents the waveguide mode under the plasmapause. Azimuthally, it is a traveling wave with a typical wavelength $\lambda_\varphi = r_p/m$. In coordinates x and z this mode is a standing wave with j nodes in coordinate x and with n nodes in coordinate z. Note that $\lambda_\varphi \ll \alpha \ll \beta$. The frequency of the eigenmode when $m \gg 1$, as follows from (14), is given by the expression

$$\omega_{mnj} = A_p\left(\frac{m}{r_p} + \frac{\xi_j m^{1/3}}{2a_p^{2/3} r_p^{1/3}} + \frac{\lambda_n}{2l_p}\right). \qquad (15)$$

The eigenfrequency is determined by three quantum numbers $m, n,$ and j. The first term in this expression is the basic term, and the other two terms describe the splitting of frequencies at a given m. With a decrease of m the scales of the eigenmode in coordinates x and z increase, and at the limit of validity of the approximation (when $m \sim 1$) they become of the order of the size of the plasmasphere r_p. The frequency of such a mode is of the order of the fundamental harmonic of the plasmasphere $A_p/r_p \sim 10^{-2}$ s^{-1}; furthermore, the three terms in (15) become of the same order of magnitude.

Let us now take into consideration that the potential barrier in Figure 3 when $x > 0$ is not infinite, and hence the energy of the eigenmode can penetrate under the barrier. This means that it is a damped mode with a certain decrement. Substituting the approximate value of $\omega^2 = m^2 A_p^2/r_p^2$ when $x > 0$ into the second relation of (5) gives

$$U(x,z) = \frac{m^2}{r_p^2}\left[1 - \frac{A_p^2}{A_{p1}^2}\left(1 + \frac{x}{a_{p1}} - \frac{z^2}{l_{p1}^2}\right)\right].$$

If this expression at a fixed z is regarded as a one-dimensional potential barrier, then the transmission coefficient through it is determined to a quasi-classical approximation as

$$D(z) = \exp\left(-\int_0^{x_c}\sqrt{U(x,z)}dx\right), \qquad (16)$$

where $x_c = x_c(z)$ is the point, at which $U(x_c, z) = 0$. By evaluating in (16) the integral in the equatorial plane, we get

$$D(0) \equiv D_0 = e^{-\sigma m}, \quad \sigma = \frac{2}{3}\frac{\left(1 - (A_p/A_{p1})^2\right)^{2/3}}{[2 + (A_p/A_{p1})^2(r_p/a_{p1})]}.$$

When $z \neq 0$, the transmission coefficient has the same form, but the value of σ is larger, i.e., $D(l) < D_0$. This is natural because the potential barrier in the equatorial plane has the least height.

The consistent theory for the underbarrier penetration from a two-dimensional potential well has not yet been developed in a general form. In the present case it is anticipated to obtain a correct estimate for the integral transmission coefficient D by averaging the local expression $D(l)$ over the wave amplitude with the aid of the relation

$$D = \frac{\int_{-\infty}^{\infty}\varphi_n^2(z/\beta)D(z)dz}{\int_{-\infty}^{\infty}\varphi_n^2(z/\beta)dz}.$$

Straightforward calculations give

$$D = \kappa D_0 = \kappa e^{-\sigma m},$$

where κ is the coefficient of order unity which has a rather unwieldy form.

The damping decrement of the mode is determined by the transmission coefficient

$$\Gamma = D\omega = \kappa\omega e^{-\sigma m}.$$

When $m \gg 1$, the decrement is exponentially small, and at the limit of applicability when $m \sim 1$ for global modes, the decrement is of the order of frequency.

4. Outer Magnetospheric Cavity Modes

We now embark on the study of the possible existence of eigenmodes bounded by an abrupt jump of the Alfven velocity on the magnetopause. At this point we examine sufficiently high frequency eigenoscillations which are narrowly localized across the magnetic shells. These oscillations represent harmonics with large azimuthal wave numbers $m \gg 1$. At small wave numbers $m \sim 1$, these oscillations change to low-frequency global modes which are localized on scales of the order of magnetospheric scales. We confine ourselves to the case c and use the following model of the Alfven velocity:

$$A = \begin{cases} A_m[1 - (x/2a_m) + (z^2/2l_m^2)] & x < 0, \\ A_{m1}[1 - (x/2a_{m1}) + (z^2/2l_{m1}^2)] & x > 0, \end{cases} \qquad (17)$$

In this section, $x = r - r_m$. In the order of magnitude, $r_m \sim 10^5$ km. The expressions (17) are applicable when $|x| \ll l_m, l_{m1}$. In the order of magnitude, $a_m, a_{m1}, l_m, l_{m1} \sim r_m$. In the validity range of the expression (17), for the potential we have

$$U = \begin{cases} (m/r_m)^2 - (\omega/A_m)^2[1 + (x/a_m) - (z/l_m)^2] \\ \qquad x < 0, \\ (m/r_m)^2 - (\omega/A_{m1})^2[1 + (x/a_{m1} - (z/l_{m1})^2] \\ \qquad x > 0, \end{cases} \qquad (18)$$

Formulas (17) and (18) are similar to the expressions (4) and (5) with the essential difference that $A_{m1} < A_m$. It will be assumed that the potential jump is sufficiently large that in the zeroth-order approximation, $U = -\infty$ can be put when $x > 0$. In this approximation,

$$U = \begin{cases} -\varepsilon - kx + gz^2 & x < 0, \\ -\infty & x > 0. \end{cases}$$

Here

$$\varepsilon = \frac{\omega^2}{A_m^2} - \frac{m^2}{r_m^2}, \quad k = \frac{m^2}{a_m r_m^2}, \quad g = \frac{m^2}{l_m^2 r_m^2}. \qquad (19)$$

The solution of this problem is quite analogous to the solution of (11)-(14):

$$\Psi = C\psi_j\left(\frac{x}{\alpha}\right)\varphi_n\left(\frac{z}{\beta}\right),$$

$$\alpha^{-1} = k^{1/3} = m^{2/3}/a_m^{1/3} r_m^{2/3},$$
$$\beta^{-1} = g^{1/4} = m^{1/2}/l_m^{1/2} r_m^{1/2},\ \varepsilon = \frac{\xi_j}{\alpha^2} + \frac{\lambda_n}{\beta^2}. \quad (20)$$

In the next order we take into consideration that when $x > 0$,

$$U = -\varepsilon_1 - k_1 x + g_1 z^2,$$

where

$$\varepsilon_1 = \frac{m^2}{r_m^2}\left(\frac{A_m^2}{A_{m1}^2} - 1\right),$$
$$k_1 = \frac{m^2}{r_m^2}\frac{A_m^2}{a_{m1}A_{m1}^2}, \qquad g_1 = \frac{m^2}{r_m^2}\frac{A_m^2}{l_{m1}^2 A_{m1}^2}.$$

We seek the solution in this region in the form

$$\Psi = f(x)\varphi_n\left(\frac{z}{\beta_1}\right).$$

Substituting into (7) gives the relation

$$\beta_1^{-1} = g_1^{1/4} = m^{1/2}A_m^{1/2}/A_{m1}^{1/2} l_m^{1/2} r_m^{1/2},$$

and the equation for the function f,

$$\frac{d^2 f}{dx^2} + \left(\varepsilon_1 - \frac{\lambda_n}{\beta_1^2} + k_1 x\right) f = 0. \quad (21)$$

Note that in the order of magnitude, $\beta \sim \beta_1$. Besides, $\varepsilon_1 \gg \lambda_n/\beta_1^2$ when $m \gg 1$.

The same approach can also be used when $x < 0$. Assuming in this region

$$\Psi = f(x)\varphi_n\left(\frac{z}{\beta}\right),$$

for β we have the previous relation (20), and for the function f we have the equation

$$\frac{d^2 f}{dx^2} + \left(\varepsilon - \frac{\lambda_n}{\beta^2} + kx\right) f = 0. \quad (22)$$

Equations (21) and (22) should be treated as equations for a common function $f(x)$, when $x > 0$ and $x < 0$, respectively. The parameter ε_1 should be considered given, and ε should be defined by the eigenvalue.

When $x \to \infty$, the solution must have the form of an escaping wave. Physically, this means that the energy escapes the region of localization of the eigenmode. Let the corresponding solution of the Airy equation, similar to (9), be designated as $\psi_+(\xi)$. It has for $\xi \gg 1$ the following asymptotic representation:

$$\psi_+(\xi) = \xi^{-1/4}\exp\left(\frac{2}{3}i\xi^{3/2}\right). \quad (23)$$

We seek the solution of (21) in the form

$$f = C_+\psi_+\left(\frac{x}{\alpha_1} + \xi_+\right),$$

which when substituted into the equation gives

$$\alpha_1^{-1} = k_1^{1/3} = m^{2/3}A_m^{2/3}/A_{m1}^{2/3}a_m^{1/3}r_m^{2/3},$$
$$\xi_+ = \alpha_1^2\left(\varepsilon_1 - \frac{\lambda_n}{\beta_1^2}\right).$$

When $m \gg 1$,

$$\xi_+ = \left(m\frac{a_m A_{m1}}{r_m A_m}\right)^{2/3}\left(\frac{A_m^2}{A_{m1}^2} - 1\right),$$

or in the order of magnitude,

$$\alpha_1 = \frac{r_m}{m^{2/3}}, \qquad \xi_+ \sim m^{2/3}.$$

Thus, when $m \gg 1$, we have $\xi_+ \gg 1$, and the asymptotic expression (23) can be used throughout the range of its applicability, i.e., when $x > 0$.

The solution of (21) that determines the function $f(x)$ when $x < 0$ is sought in the form

$$f = C_-\psi_-\left(\frac{x}{\alpha_1} + \xi_-\right), \quad (24)$$

where the function $\psi_-(\xi)$ is already defined. Substituting into the equation gives

$$\alpha^{-1} = k^{1/3} = m^{2/3}/a_m^{1/3}r_m^{2/3},$$
$$\varepsilon = \frac{\xi_-}{\alpha^2} + \frac{\lambda_n}{\beta^2}. \quad (25)$$

Note that in the order of magnitude, $\alpha \sim \alpha_1$. The value of ξ_- is determined from the condition of matching of the solutions (23) and (24) at the point $x = 0$:

$$\frac{1}{\alpha}\frac{\psi_-'(\xi_-)}{\psi_-(\xi_-)} = \frac{1}{\alpha_1}\frac{\psi_+'(\xi_+)}{\psi_+(\xi_+)}.$$

Using for the right-hand side of this relation the asymptotic expression (23), we bring it to the form

$$\psi_-(\xi_-) = -i\frac{\alpha_1}{\alpha}\xi_+^{-1/2}\psi_-'(\xi_-).$$

In the main order for the large parameter ξ_+ we have

$$\psi_-(\xi_-) = 0,$$

i.e., $\xi = \xi_j$ (zero of the function $\psi_-(\xi_-)$), and hence $\psi_-(\xi + \xi_-) = \psi_j(\xi)$. In the next order we assume $\xi_- = \xi_j + \delta$, and for δ we get

$$\delta = -i\frac{\alpha_1}{\alpha}\xi_+^{-1/2}.$$

On substituting this value for ξ_- into (25), we obtain

$$\varepsilon = \frac{\xi_j}{\alpha^2} + \frac{\lambda_j}{\beta^2} - i\frac{\alpha_1}{\alpha}\frac{1}{\xi_+^{1/2}}.$$

Substituting this expression into the first relation of (19) gives

$$\omega = \omega_{mnj} = A_m\left(\frac{m}{r_m} + \frac{\xi_j m^{1/3}}{2a_m^{2/3} r_m^{1/3}} + \frac{\lambda_n}{2l_m}\right) - i\frac{A_{m1}A_m}{2a_m\sqrt{A_m^2 - A_{m1}^2}}. \tag{26}$$

The real part of this frequency is quite analogous to the expression (15). The imaginary part gives the damping decrement of the mode caused by the escape of some of its energy to the solar wind,

$$\gamma = -\mathrm{Im}\omega_{mnj} = \frac{A_{m1}A_m}{2a_m\sqrt{A_m^2 - A_{m1}^2}},$$

which (considering that $A_m \gg A_{m1}$) is of the order of magnitude of $\gamma \sim A_{m1}/r_m$.

5. Eigenmodes in a Three-Dimensionally Inhomogeneous Cavity

As follows from (15) and (16), eigenfrequencies of the outer magnetospheric cavity modes and of the cavity under the plasmapause in an axisymmetric magnetosphere when $m \gg 1$ are proportional to m. The value of m is bounded above by the condition $\omega_{mnj} \leq \omega_i$, where ω_i is the ion-cyclotron frequency. Since the characteristic value of $A_p/r_p \sim 2 \times 10^{-2}$ s^{-1}, in the real magnetosphere $\omega_i \sim 40$ s^{-1}, and for a maximum value of m we obtain $m \leq m_{\max} \approx 2 \times 10^3$. For such large values of m the transverse size of the eigenmode is extremely small ($\alpha \sim 100$ km, $\beta \sim 500$ km). Thus the mode is localized inside a thin band encircling the Earth in the equatorial plane near the Alfven velocity minimum.

It is apparent that for such a picture the azimuthal asymmetry of the magnetosphere has a critical value and can "rupture" readily this band, bounding it azimuthally. Since the azimuthal wavelength of such oscillations is extremely small, there is a good probability that alternating transparent and opaque regions will be encountered at the azimuthal propagation. In this case each of such transparent regions may be regarded as an isolated cavity. The detection of this phenomenon would require carrying out coordinated observations using several satellites.

It is therefore of interest to examine the magnetosonic eigenoscillations localized in a three-dimensionally inhomogeneous cavity. It is a straightforward matter to generalize the picture of eigenoscillations of the axisymmetric magnetosphere proposed above, to the case of a three-dimensionally inhomogeneous magnetosphere. Assuming that the Alfven velocity distribution has a minimum not only in the radial direction but also in the azimuthal direction (at the point $y = 0$, where $y \approx r_p\varphi$), in the problem of magnetosonic eigenoscillations in the cavity under the plasmapause the following model can be used:

$$A = \begin{cases} A_p[1 - (x/2a_p) + (y^2/2b_p^2) + (z^2/2l_p^2)] & x < 0, \\ A_{p1}[1 - (x/2a_{p1}) + (y^2/2b_{p1}^2) + (z^2/2l_{p1}^2)] & x > 0. \end{cases}$$

This leads to the following expression for potential energy in the three-dimensional Schrodinger equation (1):

$$U = \begin{cases} -\varepsilon - kx + hy^2 + gz^2 & x < 0, \\ \infty & x > 0, \end{cases}$$

where

$$\varepsilon = \frac{\omega^2}{A_p^2}, \qquad k = \frac{\omega^2}{a_p A_p^2}, \qquad h = \frac{\omega^2}{b_p^2 A_p^2}, \qquad g = \frac{\omega^2}{l_p^2 A_p^2}.$$

At this point we confine ourselves to determining only the real part of the eigenfrequency and neglect the oscillation damping associated with the escape of some of the oscillation energy from the cavity.

As a result, once again, we obtain a partial differential equation with separable variables whose solution has the form

$$\Psi = C\psi_j\left(\frac{x}{\alpha}\right)\varphi_n\left(\frac{z}{\beta}\right)\varphi_M\left(\frac{y}{\gamma}\right).$$

Here $M = 0, 1, 2...$ is the quantum number in the azimuthal direction. Substituting into the equation for characteristic scales α, β, and γ gives the following expressions:

$$\alpha = \frac{a_p^{1/3}A_p^{2/3}}{\omega^{2/3}}, \qquad \beta = \frac{l_p^{1/2}A_p^{1/2}}{\omega^{1/2}}, \qquad \gamma = \frac{b_p^{1/2}A_p^{1/2}}{\omega^{1/2}},$$

and the dispersion equation is of the form

$$\varepsilon = \frac{\xi_j}{\alpha^2} + \frac{\lambda_j}{\beta^2} + \frac{\lambda_M}{\gamma^2}, \tag{27}$$

where ξ_j are the zeroes of the Airy function $Ai(\xi)$, $(j = 1, 2, 3...)$, $\lambda_n = 2n + 1$, and $\lambda_M = 2M + 1$. When $M \gg 1$, this equation leads to the following asymptotic expression for the frequency spectrum of a three-dimensional cavity:

$$\omega_{Mnj} = A_p \left(\frac{2M}{b_p} + \xi_j \frac{2^{1/3} M^{1/3}}{a_p b_p^{1/3}} + \frac{\lambda_n}{l_p} \right). \qquad (28)$$

A similar equation is also obtained for outer magnetospheric cavity modes, in which the index p denoting the parameters in the neighborhood of the plasmapause should be substituted for m denoting the corresponding parameters near the magnetopause.

6. Discussion

Recently, there has been a great deal of observational evidence for low-frequency MHD oscillations of the magnetosphere with a rather stable discrete frequency spectrum ($f \sim 1.3; 1.7; 2.1; 3.3...$ mHz). These oscillations were mostly recorded in the nightside part of the magnetosphere both at the ground-based network of observing stations [*Davidson and Orr*, 1989; *Samson et al.*, 1992; *Ziesolleck and McDiarmid*, 1994] and by satellites [*Kivelson et al.*, 1984]. Coordinated observations of these oscillations were recently carried out at the ground-based Canadian Auroral Network for the OPEN Program Unified Study (CANOPUS) and by the CRRES and GOES 6 and 7 satellites [*Lessard et al.*, 1999]. As a result of this experiment, it was established that the oscillations on the ground and at the CRRES satellite (which was near the midnight ionosphere at that time) are standing Alfven waves with a structure typical of field line resonance. The phase velocity of such oscillations was found to be directed antisunward, which suggests that their source was located in the daytime part of the magnetosphere. The GOES satellites that were at the top of the resonance field lines at the time of experiment recorded narrowly localized magnetosonic oscillations around the frequency of 2.1mHz. This suggests that these oscillations (at least at 2.1-mHz frequency) are responsible for field line resonance.

The origin of the low-frequency magnetosonic oscillations of the magnetosphere with a discrete frequency spectrum is not yet entirely known. A most natural explanation (as was indeed done initially) is the suggestion that these oscillations are eigenmodes of the magnetospheric cavity (cavity modes) [*Kivelson et al.*, 1984]. Such an approach was instrumental in explaining the origin of the low-latitude geomagnetic Pi2 pulsations [*Sutcliffe and Yumoto*, 1991]. However, simple estimates of the frequency of global magnetospheric oscillations $f \sim A/L$, where A is a typical value of the Alfven velocity and L is a typical scale of the magnetosphere, show that the frequency cannot be low enough to explain the lowest-frequency oscillations among those observed with a discrete spectrum. Indeed, we have $L \sim 15\,R_E \sim 10^5$ km and a minimum equatorial value of the Alfven velocity in the dayside undisturbed magnetosphere $A \sim 300$ km s^{-1}, which gives $f \sim 3$ mHz. More accurate calculations reported by *Lee and Lysak* [1989, 1991] and also the estimating formulas obtained in this paper give still higher values for the fundamental frequency of magnetosonic eigenocillations of the cavity.

An alternative approach to interpreting the high-latitude magnetosonic oscillations with a discrete frequency spectrum is as follows. These oscillations are treated as eigenoscillations propagating in a waveguide produced by the magnetospheric tail [*Samson at al.*, 1992; *Mann and Wright*, 1999]. However, as was with good reason pointed out by *Wright* [1994], if the length of such a waveguide is not bounded along its axis, then the frequencies of its eigenoscillations are not quantized but form a continuous spectrum. Of course, we are reminded that actually the magnetotail length is bounded (from $30\,R_E$ to $200\,R_E$ during periods of a different level of geomagnetic disturbance); in this case, however, the magnetotail should now be treated as an antisunward stretching cavity. The magnetosonic waves propagating therein may be treated as a dynamic regime of establishing eigenocillations, following their local impulsive excitation. For such a cavity the above estimates of its eigenfrequencies are applicable (in this case, L is a typical radius of the tail).

Of course, for constructing an accurate picture of magnetosonic eigenocillations of the magnetosphere, it is necessary to use a three-dimensionally inhomogeneous model of the magnetosphere. This model should include the actual form of the magnetospheric cavity, including the magnetotail. Also, it does well to bear in mind that it is unlikely that steady state regimes of oscillations in the real magnetosphere are attainable. For instance, oscillations excited in the dayside of the magnetospheric cavity will be totally damped, not only because of the possibility of their partial penetration into the solar wind through the magnetopause but also because of the dynamic escape to the tail. It seems also important that the solar wind structure near the magnetopause must be taken into account. The presence of a traveling plasma outside the magnetosphere and also the bow shock must lead to a more complicated picture of the eigenocillations. This would imply the emergence of eigenmodes with lower eigenfrequencies [see *Harrold and Samson*, 1992].

However, it is not yet possible to take into account all properties of the magnetosonic eigenocillations. Therefore the estimating formulas of eigenfrequency spectra obtained in this study may be regarded as a certain zero-order approximation. Nevertheless, these formulas permit us to analyze qualitatively the dependencies of frequency spectra on typical magnetospheric parameters and on wave numbers characterizing the spatial structure of the field of magnetosonic eigenoscillations.

7. Conclusion

The main results of this study may be summarized as follows:

1. On the basis of simple models of the Alfven velocity distribution in the magnetosphere, a qualitative analysis was made of the spatial structure of magnetosonic eigenoscillations of the magnetosphere.

2. The problem of the eigenmode structure in the cavity under the plasmapause has been solved in terms of a simple axisymmetric model of the magnetosphere. An asymptotic (when $m \gg 1$) expression (equation (15)) was obtained for the eigenfrequency spectrum of such a cavity. It has been shown that the oscillations in such a cavity are attenuated because of the escape of some of their energy to the outer magnetosphere.

3. The spatial structure of the outer magnetospheric cavity modes has been investigated within the framework of a simple axisymmetric magnetosphere. An asymptotic (when $m \gg 1$) expression (equation (26)) was obtained for the spectrum of mode eigenfrequencies. These oscillations are also damped ones, because the magnetosphere is not a perfectly reflecting boundary, and some energy of the modes escapes to the solar wind region.

4. The problem of the structure of magnetosonic eigenoscillations in a three-dimensional magnetospheric cavity has been solved. The dispersion equation (27) was obtained for the eigenfrequencies of this cavity. An explicit expression (equation (28)) for the eigenfrequency spectrum was obtained in the limit of large azimuthal eigennumbers.

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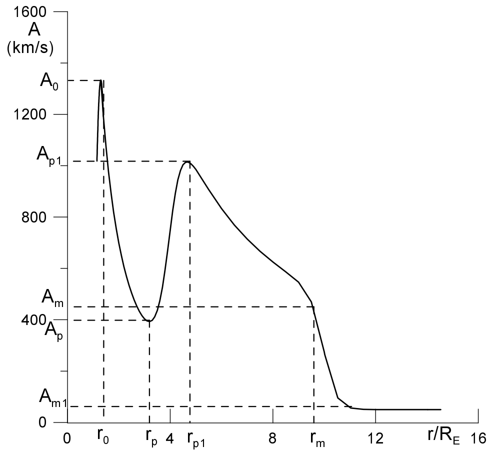

Figure 1. Typical Alfven velocity profile in the equatorial section of the dayside magnetosphere. Here r_m is the magnetopause radius, r_p is the radius of the plasmapause, and r_0 is the radius of maximum $A(r, 0)$ on the magnetic shell L=1.3.

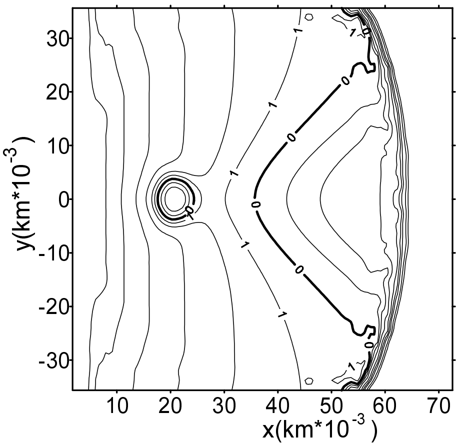

Figure 2. Level lines of the potential $u(x, y)$ in meridional plane in the dipole magnetosphere, corresponding to case b.

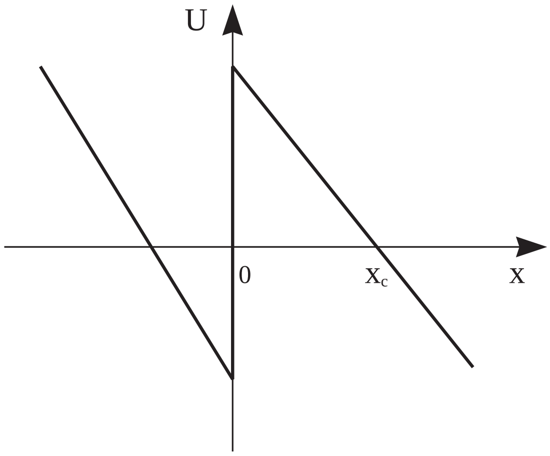

Figure 3. Equatorial section of the model potential $U(x, 0)$ in equation (5), corresponding to the cavity for magnetosonic oscillations under the plasmapause.