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RESONANCE EXCITATION OF STANDING ALFVEN WAVES IN AN AXISYMMETRIC MAGNETOSPHERE (MONOCHROMATIC OSCILLATIONS)

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Abstract—A theoretical study is made of the phenomenon of standing Alfven wave resonance excitation by a monochromatic fast magnetosound in an axisymmetric model of the magnetosphere. A theory is developed which includes the Alfven and magnetosound wave relationship, transverse and longitudinal inhomogeneities of the medium, the weak transverse dispersion of Alfven waves and their dissipation on the ionosphere. General formulae are derived which describe the spatial structure of a monochromatic Alfven wave. These formulae are specified for all physically differing cases of the behaviour of magnetospheric parameters. An analysis is made of the back influence of an Alfven wave on a magnetosound field.

1. INTRODUCTION

Standing Alfven waves in the Earth's magnetosphere have been the subject of theoretical study since the appearance of a fundamental paper of Dungey (1954). The extreme complexity of this problem forces researchers to restrict themselves to very simplified problem statements. In the development of a theory of standing waves (see a review by Southwood and Hughes, 1983 and references therein) one may note the following two stages.

On the basis of equations of ideal magnetic hydrodynamics and in a model of an axisymmetric (i.e. two-dimensionally inhomogeneous—from the mathematical point of view) magnetosphere, Radoski (1967), Radoski and Carovillano (1969) and Cummings *et al.* (1969) considered two particularly simple types of standing Alfven waves which are Alfven eigen-oscillations. These are axisymmetric, toroidal modes for which $m = 0$, where m is the azimuthal wave number, and poloidal modes with unlimitedly large values of $m \to \infty$. In these two limits the system of magnetohydrodynamical equations splits into independent equations for Alfven waves and a fast magnetosound, and this makes the study drastically simpler. The solutions for Alfven waves have a remarkable feature, namely they are concentrated on separate magnetic resonance shells for toroidal modes and on separate field-lines lying on resonance shells, for poloidal modes, i.e., their eigen-functions are δ-functions of transverse coordinates. Toroidal modes can be treated as oscillations of separate shells and the poloidal modes as separate lines of force. The field-aligned (along the geomagnetic field) structure of such waves is described by a one-dimensional differential equation which, together with the boundary conditions on an ideally conducting ionosphere, leads to a one-dimensional problem for eigen-values. This determines the set of field-aligned eigen-modes (harmonics) and eigen-frequencies. Taking account of finite conductivity of the ionosphere leads to the damping of the eigen-modes (Newton *et al.*, 1978).

Toroidal and poloidal modes are very partial solutions of the equations but, owing to their obviousness, are widely used when interpreting observational data. During the past decade standing waves have been intensively studied with the aid of spacecraft (see Takahashi and McPherron, 1984 and references therein), which made it possible to collect a large amount of experimental evidence. Unlike ground-based observational data, this information is free from the masking effect of the ionosphere, atmosphere and the Earth's surface. They demonstrated convincingly the existence of standing waves, their harmonic structure and good agreement with the theoretical frequency spectrum of toroidal and poloidal modes.

However, in the theoretical papers cited above the question of the excitation mechanism for standing waves remains unanswered. The second stage of theoretical investigations that deserves mention here implies constructing a resonance theory of Alfven wave excitation in the magnetosphere as initiated in papers of Southwood (1974) and Chen and Hasegawa (1974). According to this theory, an Alfven wave is excited as a result of so-called Alfven resonance. Such resonance requires a relationship between the waves of both types which exists only when values of m are finite and different from zero. In this case, however, the problem, even in the ideal MHD approximation, becomes greatly complicated. Therefore, the papers mentioned above employed an extremely simplified model of the medium in the form of a flat plasma sheet. It is supposed that the geomagnetic field is a homogeneous one and the plasma density depends only on one transverse coordinate. The nonzero value of m is modelled by the wave vector component in the direction perpendicular to the magnetic field and to the density gradient. The position of the resonance surface (in this model it is a plane) is determined by the condition of equality of the local Alfven frequency and the magnetosound frequency.

Papers of Southwood (1974) and Chen and Hasegawa (1974), together with a number of subsequent ones, constitute a considerable part of the theory of Alfven resonance in a one-dimensionally inhomogeneous plasma (see a review by Stix and Swanson, 1980 and references therein). In the ideal MHD approximation the disturbance field has on the resonance surface a singularity which corresponds, in a sense, to that of eigen-functions of toroidal and poloidal modes. Taking account of effects outside the framework of ideal MHD eliminates this singularity. Such effects are the dissipation in the ionosphere as well as the effect of Larmor radius of ions and the electron inertia. These latter lead to transverse dispersion of Alfven waves which are called kinetic waves in this case (Hasegawa, 1976; see also Goertz, 1984).

Recently there emerged a challenging problem of constructing a theory for Alfven resonance in non-one-dimensional models of the magnetosphere. There are already a number of papers devoted to this question (Krylov *et al.*, 1981; Krylov and Lifshitz, 1984; Southwood and Kivelson, 1986). These papers have a limitation in common, namely that it is impossible to use the formulae obtained for specific applications. They establish some qualitative properties of the solution of MHD equations and, in particular, the conclusion about the presence of a resonance surface is drawn and the solution singularity is investigated near it. However, mathematical difficulties encountered when solving non-one-dimensional equations remain unresolved; the explicit form of these solutions has not been found.

The present paper is devoted to the theory of Alfven resonance in an axisymmetric model of the magnetosphere. The result obtained involves formulae which totally define the spatial structure of the field of a monochromatic Alfven wave. The formulae bring about some kind of synthesis of the theory of Alfven resonance in a flat sheet with the theory of toroidal mode in an axisymmetric magnetosphere; they reduce the solution of the two-dimensional problem to a consecutive solution of two one-dimensional problems, and these latter can be solved explicitly in most practically important cases. In particular, the success of the toroidal mode theory in determining the frequency spectrum of standing waves finds its explanation and a rigorous mathematical substantiation is given for the use of the flat sheet model.

2. FORMULATION OF THE PROBLEM

In our treatment of the magnetosphere we shall be using a curvilinear orthogonal coordinate system x^1, x^2, x^3 in which surfaces $x^1 = \text{const.}$ coincide with magnetic shells, the coordinate x^2 specifies the line of force on a given shell and x^3 specifies a point on a given line of force (see Fig. 1). The system is assumed axially symmetric; this is the only constraint we impose on the model magnetosphere. It is of importance theoretically because it changes the problem from a three-dimensionally inhomogeneous (as it actually is) over to a two-dimensionally inhomogeneous problem, which significantly simplifies mathematical developments. At the same time the two-dimensionally inhomogeneous model is much more tenable as compared with the usually used one-dimensional model and, in the known approximation, describes adequately the inner part of the magnetosphere, which cannot be said about the one-dimensional model. It should be stressed that symmetry about the equator is not assumed. For an axially

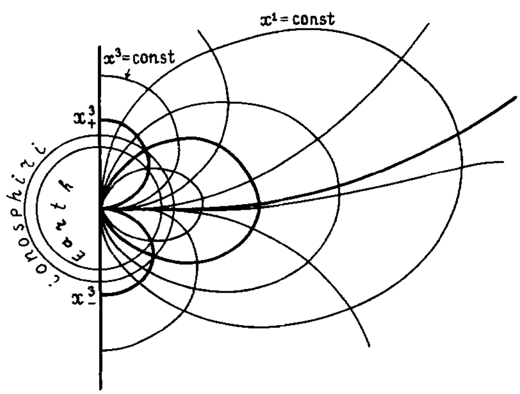

FIG. 1. A CURVILINEAR, ORTHOGONAL SYSTEM OF COORDINATES (x^1, x^3) IN THE MERIDIONAL PLANE ($x^2 = \text{const.}$).
Emphasis is deliberately placed on the possible North–South asymmetry of the magnetosphere. The figure singles out: the equatorial line that is a separatrix for coordinate curves $x^3 = \text{const.}$, one of the magnetic shells and coordinate lines $x^3 = \text{const.}$ corresponding to the intersection of this shell with the ionosphere in the Northern and Southern (x^3_+ and x^3_-) Hemispheres.

symmetric system it is natural to use as the coordinate x^2 the azimuthal angle φ. We denote by x^3_+ and x^3_- the coordinates of intersection of the line of force with the ionospheres of the conjugate hemispheres; they are functions of the magnetic shell: $x^3_\pm = x^3_\pm(x^1)$.

Disturbances in a wave will be assumed by covariant components of the disturbed magnetic field vector, and we shall operate on both the components themselves and their Fourier-harmonics

$$B_i(x^1, x^2, x^3, t) = \int_{-\infty}^{\infty} \tilde{B}_i(x^1, x^2, x^3, \omega)\, e^{-i\omega t}\, d\omega.$$

Taking into account the axial symmetry of the system the dependence of a disturbance can be chosen in the form exp (ik_2x^2), where k_2 is a covariant azimuthal component of the wave vector. When $x^2 = \varphi$, then $k_2 = m$, where m is the azimuthal wave number. In fact, we shall not need such a representation, but for the estimations we shall be using the representation of a certain typical mean value of $k_2 \sim \partial/\partial x^2$ (or a corresponding value of m). An important factor deserves mention here. If an azimuth-bounded disturbance is considered, then the axial symmetry condition can be regarded as satisfied, provided that the parameters of the medium are independent of azimuth φ within the region in which the disturbance is localized. Such a requirement is a less stringent one, i.e., in the dayside sector it can be considered roughly satisfied on magnetic shells as far as the magnetopause.

Similar to k_2 we shall also be treating the other components of the wave vector: $k_1 \sim \partial/\partial x^1$ and $k_3 \sim \partial/\partial x^3$. Note that relevant physical components, i.e., those in a local Euclidian basis, are given by the relation $\hat{k}_i = k_i/\sqrt{g_i}$, where g_i represents diagonal components of a metric tensor.

The standing Alfven waves of interest pertain to a class of so-called quasi-perpendicular Alfven waves whose characteristic values of the transverse and the longitudinal wave vectors satisfy the relation $k_\perp \gg (\omega/\omega_i)^{1/2}k_\parallel$, where $\omega_i = eB_0/m_ic$ is the cyclotron frequency of ions. Quasi-perpendicular Alfven waves have a nearly linear polarization in a plane normal to the external magnetic field. In this paper we restrict ourselves to a consideration of the oscillations with not too great values of azimuthal wave number m such that for an Alfven wave the condition

$$\hat{k}_2 \ll \hat{k}_1 \tag{1}$$

is satisfied. Such waves are close to toroidal modes. In this case the Alfven wave can be described by the component B_2 and the magnetosound wave can be represented by the component B_3. On the order of magnitude we have $\hat{k}_1 \sim 1/\mathscr{C}$, where $\mathscr{C}$ is a typical scale of the solution for the coordinate x^1, and $\hat{k}_2 \sim m/L$, where L is the length of the line of force. The condition formulated above can be rewritten as $m \ll L/\mathscr{C}$. It is not too stringent because, as will be shown below, $\mathscr{C} \ll L$ (a typical value of $L/\mathscr{C} \sim 10^2$). Moreover, that this condition is satisfied is actually provided by the excitation mechanism for an Alfven wave.

In this paper we adopt the hypothesis according to which a magnetosound is generated by an external source and penetrates, then, into the magnetosphere. Such a source may be provided by an instability of a proton flux reflected from the bow shock front (Gul'elmi, 1984) or the Kelvin–Helmholtz instability on the magnetopause (Kivelson and Pu, 1984). Simple reasoning shows that on a resonance surface the magnetosound is located in the opacity region, i.e., does not oscillate but drops exponentially along the coordinate x^1 towards the Earth. A typical scale of such a decline is of the order L/m and for the magnetosound wave might be able to penetrate sufficiently deep into the magnetosphere, the value of m must not be too large ($m \lesssim 10$). This does not, however, rule out the possibility that the magnetosound may have an oscillatory character in the outer magnetosphere.

Disturbed fields of a monochromatic wave obey the equations

$$\text{curl}\, \mathbf{E} = i\frac{\omega}{c}\mathbf{B}, \quad \text{curl}\, \mathbf{B} = -i\frac{\omega}{c}\hat{\varepsilon}\mathbf{E}, \tag{2}$$

where $\hat{\varepsilon}$ is the dielectric permittivity tensor. For the oscillations of our interest, it can be considered diagonal. Physical components of the tensor are given in a paper by Akhiezer *et al.* (1974)

$$\varepsilon_{11} = \frac{c^2}{A^2}(1 - \tfrac{3}{4}k_\perp^2\rho_i^2), \quad \varepsilon_{22} = \frac{c^2}{A^2}\left(1 - 2\frac{\omega_i^2}{\omega^2}k_\perp^2\rho_i^2\right),$$

$$\varepsilon_{33} = \frac{c^2}{\omega^2\Lambda_s^2}. \tag{3}$$

Here $A = B_0/\sqrt{4\pi m_in}$ is the Alfven velocity, $\rho_i = v_i/\omega_i$ is the Larmor radius of ions, the remaining notations are:

$$\Lambda_s^2 = \frac{\rho_s^2}{w(\omega/k_\parallel v_e)}, \quad \rho_s = \frac{v_s}{\omega_i}, \quad v_s = \left(\frac{T_e}{m_i}\right)^{1/2},$$

$$v_e = \left(\frac{T_e}{m_e}\right)^{1/2}, \quad v_i = \left(\frac{T_i}{m_i}\right)^{1/2},$$

and $w(z)$ is the (known in plasma physics) function

$$w(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{t\,e^{-t^2/2}\,dt}{t-z} = 1 - z\,e^{-z^2/2} \int_0^z e^{t^2/2}\,dt + i\left(\frac{\pi}{2}\right)^{1/2} z\,e^{-z^2/2}. \quad (4)$$

For real z it has the following limiting expressions:

$$w(z) = \begin{cases} 1 - z^2 + \cdots + i(\pi/2)^{1/2} z, \\ \quad |z| \ll 1, \\ -\dfrac{1}{z^2} - \dfrac{3}{4z^4} + \cdots + i\left(\dfrac{\pi}{2}\right)^{1/2} z\,e^{-z^2/2}, \\ \quad |z| \gg 1. \end{cases}$$

In the approximation of ideal MHD $\rho_i = \Lambda_s = 0$. In this case, in a homogeneous plasma, the relations (2) and (3) describe the independent magnetosound and Alfven waves with the dispersion laws, respectively, $\omega^2 = k^2 A^2 = (k_\parallel^2 + k_\perp^2) A^2$ and $\omega^2 = k_\parallel^2 A^2$. A distinctive feature of the Alfven wave is the absence of transverse dispersion, namely the frequency ω does not depend on the transverse wave vector $k_\perp$. When leaving the framework of ideal MHD the Alfven wave shows a weak transverse dispersion which is described for quasi-perpendicular waves by a simple formula

$$\omega^2 = k_\parallel^2 A^2 (1 + k_\perp^2 \Lambda^2), \quad (5)$$

where

$$\Lambda^2 = \Lambda_s^2 + \tfrac{3}{4}\rho_i^2. \quad (6)$$

Here and in what follows, it is assumed that $k_\perp \Lambda \ll 1$. Such waves were called the kinetic Alfven waves (Hasegawa and Uberoi, 1982). Even if dispersion is taken into account, the approximate equality $\omega \simeq k_\parallel A$ remains valid so that in the argument of the function w one can put

$$\frac{\omega}{k_\parallel v_e} = \frac{A}{v_e} \equiv \frac{s}{\rho_s} \equiv \frac{m_e/m_i}{\beta_e},$$

where $s = c/\omega_{pe}$ is the electron inertial (skin) length and $\beta_e = 8\pi n_0 T_e / B_0^2$ is the electron to magnetic pressure ratio. One may write

$$\Lambda_s^2 = \frac{\rho_s^2}{w(s/\rho_s)}. \quad (7)$$

Specifically, we have

$$\Lambda_s^2 = \begin{cases} -s^2 & s \gg \rho_s; \quad \beta_e \ll m_e/m_i, \\ \rho_s^2, & s \ll \rho_s; \quad \beta_e \gg m_e/m_i. \end{cases}$$

When $s \sim \rho_s$ (i.e., $\beta_e \sim m_e/m_i$), the value of Λ_s^2 is complex, and $\mathrm{Im}\,\Lambda_s^2 < 0$. As is apparent in (5), this corresponds to the damping of the wave. The physical mechanism for such a damping is Cherenkov resonance on electrons.

In an inhomogeneous, axially symmetric plasma, from (2) and (3) one may derive the system of equations

$$\frac{g_2}{\sqrt{g}} \frac{\partial}{\partial x^3} A^2 \frac{g_1}{\sqrt{g}} \frac{\partial \tilde{B}_2}{\partial x^3} + \omega^2 \tilde{B}_2 + \frac{g_2}{\sqrt{g}} \frac{\partial}{\partial x^1} \omega^2 \Lambda^2 \frac{g_3}{\sqrt{g}} \frac{\partial \tilde{B}_2}{\partial x^1} = \frac{g_2}{\sqrt{g}} \frac{\partial}{\partial x^3} A^2 \frac{g_1}{\sqrt{g}} \frac{\partial \tilde{B}_3}{\partial x^2}, \quad (8a)$$

$$\frac{g_3}{\sqrt{g}} \frac{\partial}{\partial x^1} A^2 \frac{g_2}{\sqrt{g}} \frac{\partial \tilde{B}_3}{\partial x^1} + \frac{A^2}{g_2} \frac{\partial^2 \tilde{B}_3}{\partial x^{2^2}} + \frac{A^2}{g_1} \frac{\partial}{\partial x^3} \frac{g_1}{\sqrt{g}} \frac{\partial}{\partial x^3} \frac{\sqrt{g}}{g_3} \tilde{B}_3 + \omega^2 \tilde{B}_3 = \frac{A^2}{g_2} \left(\frac{\partial}{\partial x^3} \ln \frac{g_2}{g_1}\right) \frac{\partial \tilde{B}_2}{\partial x^2}. \quad (8b)$$

Here $g = g_1 g_2 g_3$ is a determinant of the metric tensor. The parameter Λ^2 has the same definition (6) and (7) as in the case of a homogeneous plasma, but now it is a function of coordinates: $\Lambda^2 = \Lambda^2(x^1, x^3)$.

If the oscillations are axially symmetric, $\partial/\partial x^2 = 0$, then the right-hand sides of equations (8) go to zero. In this case equation (8a) describes a toroidal Alfven mode, while (8b) represents a magnetosound wave that is independent of it. In order to avoid misunderstanding, we wish to note the following. In the flat layer model with straight geomagnetic field-lines the right-hand side of (8b) disappears. Actually, it has terms which remain in this limit. But they are omitted in (8b) because—with a really existing curvature of the lines of force—they are small as compared with the term introduced.

If the oscillations are not axially symmetric, then the Alfven wave and the magnetosound show a relationship specified by the right-hand sides of equations (8). The Alfven wave influence upon magnetosound will be examined in Section 9. It turns out that it is universal in character such that equations for magnetosound can be solved independently of the equation for an Alfven wave. For realistic models of the medium, the problem of magnetosound can only be solved numerically and should be the subject of a separate study. We shall consider the magnetosound wave field to be given. As a result, the problem for the Alfven wave is one of solving equation (8a) in which the field B_3 plays the role of a source.

Equation (8a) should be supplemented with boundary conditions on the ionosphere. A large number of papers (see Southwood and Hughes, 1983 and references therein) are devoted to their derivation. Their results can be represented as

$$\left(\frac{\partial \tilde{B}_2}{\partial x^3} \mp i\delta_{\pm} \frac{\omega\sqrt{g_3}}{A} \tilde{B}_2\right)_{x^3=x^3_{\pm}} = 0. \quad (9)$$

Here the "+" and "−" signs correspond to the conjugate ionospheres, with

$$\delta_{\pm} = \frac{c^2 \cos \chi_{\pm}}{4\pi \Sigma_p^{(\pm)} A_{\pm}}, \quad (10)$$

where χ is the angle between the normal to the ionosphere and the line of force, Σ_p is the integral Pedersen conductivity of the ionosphere, and $A_{\pm} = A(x^1, x^3_{\pm})$ is the value of Alfven velocity on the conventional boundary between the ionosphere and the magnetosphere.

Despite the fact that condition (9) is a well-known one, it is necessary to make some comments on it. The point here is that above the layer of increased Hall and Pedersen conductivity there is the collisionless upper ionosphere where the Alfven velocity varies very rapidly with height, from a minimum value of 300–500 km s^{-1} in the $F2$ layer to a maximum value of $(1–3) \times 10^4$ km s^{-1} at the height of $(1.5–2) \times 10^3$ km. A question arises: Where must the boundary between the ionosphere and the magnetosphere be drawn and, accordingly, what values of $A_{\pm}$ must be involved in formulae (9) and (10)? In papers dealing with the derivation of the boundary conditions it is assumed that the conducting ionospheric layer comes into contact with the homogeneous magnetosphere. It is clear that in such a model the question formulated is impossible to solve. There is still another factor which does not permit the derivation of the relationship (9) to be accepted directly. The point here is that the papers in question considered Alfven oscillations whose wavelength is much larger than the typical thickness of the ionospheric layer. In our case the situation is the opposite, namely the typical transverse length of a monochromatic Alfven wave near the ionosphere is much less than the thickness of this layer. The boundary condition for such transversely small-scale Alfven waves was considered in a paper of Hughes and Southwood (1976), but only for a perpendicular geomagnetic field.

We have investigated the questions involved, and a detailed discussion will be published elsewhere. The results obtained may be summarized thus. The boundary condition for transversely small-scale Alfven waves and for an oblique geomagnetic field retains the form (9), despite the fact that concrete details of the calculations are altered substantially. The boundary between the ionosphere and the magnetosphere should be drawn at a height corresponding to the Alfven velocity maximum, i.e., formulae (9) and (10) must involve precisely this maximum velocity. The inference just made appears quite a natural one. Above $(1.5–2) \times 10^3$ km the magnetospheric parameters vary slowly and the magnetosphere can be considered—within the required accuracy—homogeneous. On the other hand, $k_{\parallel}\Delta \ll 1$, where $\Delta \sim 10^3$ km is the thickness of the upper ionosphere; therefore its presence does not affect the form of the boundary conditions.

The smallness of the dimensionless parameters $\delta_{\pm}$ characterizes the weakness of dissipation in the ionosphere. In the case of an ideally conducting ionosphere $\Sigma_p \to \infty$ and $\delta_{\pm} \to 0$. A typical physical value (for daytime conditions) is $\delta_{\pm} \sim 10^{-2}$.

3. FIELD-ALIGNED EIGEN-MODES

In what follows we shall base our discussion on the solution of an auxiliary problem for eigen-values

$$\hat{R}(\omega)H \equiv \frac{g_2}{\sqrt{g}} \frac{\partial}{\partial x^3} A^2 \frac{g_1}{\sqrt{g}} \frac{\partial H}{\partial x^3} + \omega^2 H = 0,$$

$$\left.\frac{\partial H}{\partial x^3}\right|_{x^3=x^3_{\pm}} = 0. \quad (11)$$

The coordinate x^1 plays here the role of a parameter on which the eigen-values and the eigen-functions

$$\omega = \Omega_N(x^1), \quad H = H_N(x^1, x^3), \quad (12)$$

depend, where $N = 1, 2, \ldots$ is the harmonic number. Because the problem is a Hermitian one, the frequencies Ω_N are real. From general considerations follows the completeness of the system of functions H_N (as functions of the variable x^3) as well as the possibility of choosing them as orthonormalized with an appropriate weight:

$$\oint \frac{\sqrt{g}}{g_2} H_N H_{N'} \, dx^3 \equiv 2 \int_{x^3_-}^{x^3_+} \frac{\sqrt{g}}{g_2} H_N H_{N'} \, dx^3 = \delta_{NN'}.$$

The sign of the curvilinear integral implies integrating over the line of force between the magneto-conjugate ionospheres "forward and back".

The problem for eigen-values (11) can be regarded as the limit of the problem (8a) and (9) for $\Lambda^2 = 0$, $\delta_{\pm} = 0$ and $\partial/\partial x^2 = 0$. In other words, equations (11) describe the longitudinal structure of toroidal eigen-modes. Their full (two-dimensionally inhomogeneous) spatial structure is determined by the relation

$$\tilde{B}_2(x^1, x^3) = C\delta(x^1 - \bar{x}^1) H_N(x^1, x^3), \quad (13)$$

where C is an arbitrary constant, and $\bar{x}^1$ is a resonance magnetic surface, on which the mode is concentrated. The frequency of this mode $\omega = \Omega_N(\bar{x}^1)$.

For fundamental harmonics ($N \sim 1$), the problem (11) can only be solved numerically (Radoski, 1967; Cummings *et al.*, 1969). But for higher harmonics ($N \gg 1$), the WKB approximation is applicable. In this case we have

$$\Omega_N(x^1) = N\Omega(x^1), \quad \Omega(x^1) = 2\pi/t_A(x^1),$$

$$H_N(x^1,x^3) = \left(\frac{2}{At_A}\right)^{1/2}\left(\frac{g_2}{g_1}\right)^{1/4} \times \cos\left[\Omega_N \int_{x^3_-}^{x^3} \frac{\sqrt{g_3}\,dx^3}{A}\right], \quad (14)$$

where

$$t_A(x^1) = \oint \frac{\sqrt{g_3}\,dx^3}{A},$$

is the transit time with Alfven velocity along the line of force forward and back. Note that these formulae describe qualitatively the solution even for $N \sim 1$.

Dissipation in the ionosphere can be taken into account by using the same (one-dimensional with respect to the coordinate x^3) approach. For this purpose we utilize the perturbation theory in small parameters $\delta_\pm$. Formulae (12) give a zeroth approximation. In the next order we put $H = \bar{H}_N = H_N + h_N$, where $h_N = h_N(x^1, x^3)$ is a first-order correction. Linearization of the boundary condition (9) yields

$$\left(\frac{\partial h_N}{\partial x^3} \mp i\delta_\pm \frac{\omega\sqrt{g_3}}{A} H_N\right)_{x^3 = x^3_\pm} = 0. \quad (15)$$

We linearize equation (11) bearing in mind that the difference $(\omega^2 - \Omega_N^2)$ is a value of the first order of smallness. By multiplying the relation by $(\sqrt{g}/g_2)H_N$ and integrating along the line of force, we get

$$\omega^2 - \Omega_N^2 = -\oint H_N \hat{R}(\Omega_N) h_N \, dx^3.$$

On calculating the last integral with the help of the boundary condition (15), we have

$$\oint H_N \hat{R}(\Omega_N) h_N \, dx^3 = 2i\omega\gamma_N, \quad (16)$$

where

$$\gamma_N = \gamma_N(x^1) = \delta_+ \left[\left(\frac{g_1}{g_2}\right)^{1/2} AH_N^2\right]_{x^3_+} + \delta_- \left[\left(\frac{g_1}{g_2}\right)^{1/2} AH_N^2\right]_{x^3_-}.$$

Hence we have $\omega = \pm\Omega_N - i\gamma_N$, i.e., the quantity γ_N is the damping decrement of the toroidal mode. For $N \gg 1$, when formulae (14) are applicable, we have

$$\gamma_N = \gamma(x^1) = \pi^{-1}[\delta_+(x^1) + \delta_-(x^1)]\Omega(x^1).$$

In this approximation the damping decrement does not depend on number N.

The explicit expression for the correction h_N will then not be needed, so we shall drop the procedure of calculating it.

4. TWO-DIMENSIONAL SPATIAL STRUCTURE OF A MONOCHROMATIC STANDING ALFVEN WAVE

We shall solve equation (8a) with the boundary condition (9), by further developing the perturbation theory in the small parameters Λ^2 and $\delta_\pm$. A particularly important role is played in the subsequent discussion by the smallness of Λ^2. Or more exactly, if we introduce a typical scale l along the coordinate x^1 and a scale L along the coordinate x^3 (this latter being simply the length of the line of force) and make, using them, the coordinates x^1 and x^3 dimensionless, then the ratio of the coefficients with second-order derivatives

$$\varepsilon = \frac{\omega\Lambda}{A}\frac{L}{l} \equiv k_\parallel L \frac{\Lambda}{l} \ll 1 \quad (17)$$

should be considered to be the small parameter. For fundamental harmonics ($N \sim 1$), we have $k_\parallel \sim 1/L$ and, hence, $\varepsilon \sim \Lambda/l$. A simple physical meaning can be imparted to the parameter ε. Let t_A be the transit time along the line of force with the Alfven velocity, and t_g be the displacement time of the Alfven wave perpendicularly to the magnetic shells (due to the presence of a small transverse group velocity) at typical distances l. Then $\varepsilon = t_A/t_g$ and the inequality $\varepsilon \ll 1$ means that a typical time of displacement of the wave perpendicularly to the magnetic shells is much larger than its bounce-period.

In the zeroth order in the small parameters ε and $\delta_\pm$ we have an equation and a boundary condition which describe the excitation of an Alfven wave in the ideal MHD approximation

$$\hat{R}(\omega)\tilde{B}_2 = \frac{g_2}{\sqrt{g}}\frac{\partial}{\partial x^3} A^2 \frac{g_1}{\sqrt{g}}\frac{\partial \tilde{B}_3}{\partial x^2}, \quad \left.\frac{\partial \tilde{B}_2}{\partial x^3}\right|_{x^3 = x^3_\pm} = 0. \quad (18)$$

In virtue of completeness of the system H_N any function satisfying the boundary condition (18) can be represented as a series

$$\tilde{B}_2(x^1,x^2,x^3,\omega) = \sum_N \tilde{F}_N^{(0)}(x^1,x^2,\omega)H_N(x^1,x^3), \quad (19)$$

whose coefficients depend on the coordinates x^1, x^2 and on frequency ω (index "0" denotes the zeroth order). We substitute this expression into equation (18), multiply by $(\sqrt{g}/g_2)H_N$ and integrate over x^3. As a result, we obtain

$$\tilde{F}_N^{(0)}(x^1,x^2,\omega) = \tilde{\mu}_N(x^1,x^2,\omega)\left[\frac{\omega^2}{\Omega_N^2(x^1)}-1\right]^{-1}, \quad (20)$$

where

$$\tilde{\mu}_N(x^1,x^2,\omega) = \frac{1}{\Omega_N^2(x^1)}\oint H_N\frac{\partial}{\partial x^3}\left(A^2\frac{g_1}{\sqrt{g}}\frac{\partial \tilde{B}_3}{\partial x^2}\right)\mathrm{d}x^3.$$

Assuming that the magnetosound amplitude on the ionosphere is negligibly small (in virtue of opaqueness of the magnetosphere for it, this appears quite natural), we reduce (20) to the form

$$\tilde{\mu}_N(x^1,x^2,\omega) = \oint e_N(x^1,x^3)\frac{\partial \tilde{B}_3(x^1,x^2,x^3,\omega)}{\partial x^2}\mathrm{d}x^3;$$

$$e_N(x^1,x^3) = \frac{1}{\Omega_N^2}\frac{g_1}{\sqrt{g}}A^2\frac{\partial H_N}{\partial x^3}. \quad (21)$$

Note that the function $e_N(x^1,x^3)$ is, to within the term that depends only on x^1, proportional to a perturbed electric field of the toroidal mode.

The relations (19), (20) and (21) solve, in the approximation of ideal magnetic hydrodynamics, the problem of Alfven resonance in a two-dimensionally inhomogeneous system. Field singularity, characteristic for this approximation, occurs on resonance magnetic surfaces $x^1 = x_N^1(\omega)$ defined by the equation $\Omega_N^2(x_N^1) = \omega^2$. The function $\tilde{\mu}_N$ characterizing the effectiveness of resonance represents some kind of Nth Fourier harmonic of the function $\partial\tilde{B}_3/\partial x^2$.

In the next (first) order, in the parameters ε and $\delta_\pm$ we represent the solution as a series

$$\tilde{B}_2(x^1,x^2,x^3,\omega) = \sum_N \tilde{F}_N(x^1,x^2,\omega)\bar{H}_N(x^1,x^3) = \sum_N \tilde{F}_N(x^1,x^2,\omega)[H_N(x^1,x^3) + h_N(x^1,x^3)], \quad (22)$$

which ensures satisfying the boundary condition (9). The above result of the zeroth approximation allows us to state that the functions $\tilde{F}_N$ are concentrated near resonance surface $x^1 = x_N^1(\omega)$, although, as will be shown later, it is dispersion and dissipation that eliminate singularity. Two consequences follow from this consideration. Firstly, near the surface x_N^1, only one, the Nth term, is important in the sum (22). Secondly, the dependence of the function $\tilde{F}_N$ on x^1 is much stronger as compared with H_N and it is sufficient, when calculating the derivative with respect to x^1, to differentiate only the term $\tilde{F}_N$. Thus, near the surface $x^1 = x_N^1$ and from (8a) we obtain

$$\hat{R}(\Omega_N)h_N\tilde{F}_N + (\omega^2-\Omega_N^2)H_N\tilde{F}_N + \frac{\Omega_N^2\Lambda^2}{g_1}H_N\frac{\partial^2\tilde{F}_N}{\partial x^{1^2}} = \frac{g_2}{\sqrt{g}}\frac{\partial}{\partial x^3}A^2\frac{g_1}{\sqrt{g}}\frac{\partial\tilde{B}_3}{\partial x^2}.$$

We multiply this relation by $(\sqrt{g}/g_2)H_N$, integrate over x^3 and take account of equation (16). As a result, we obtain the equation for $\tilde{F}_N$:

$$\sigma_N(x^1)\frac{\partial^2\tilde{F}_N}{\partial x^{1^2}} + \left[\frac{(\omega+\mathrm{i}\gamma_N(x^1))^2}{\Omega_N^2(x^1)}-1\right]\tilde{F}_N = \tilde{\mu}_N(x^1,x^2,\omega), \quad (23)$$

where

$$\sigma_N(x^1) = \oint\Lambda^2\frac{g_3}{\sqrt{g}}H_N^2\,\mathrm{d}x^3.$$

For $N \gg 1$, this expression is reduced to the form

$$\sigma_N(x^1) = \sigma(x^1) \equiv \frac{1}{t_A}\oint\frac{\Lambda^2}{g_1}\frac{\sqrt{g_3}\,\mathrm{d}x^3}{A}.$$

In this limit the quantity $\sigma(x^1)$ is a mean along the field-line from the dispersion parameter Λ^2/g_1. The function $\sigma_N(x^1)$ is, generally speaking, a complex one, with Im $\sigma_N \leqslant 0$. We put $\sigma_N(x^1) = \rho_N^2(x^1)\,\mathrm{e}^{-\mathrm{i}\alpha_N(x^1)}$. The quantity ρ_N has a dimensionality of length, and the phase α_N lies within the interval $(0,\pi)$. In a similar manner, we put $\sigma(x^1) = \rho^2(x^1)\exp(-\mathrm{i}\alpha(x^1))$.

Equation (23) defines the transverse structure of the Alfven wave. It is one-dimensional (the coordinate x^2 enters in it as a parameter). This equation is totally identical to that of a one-dimensional model. In this sense one can believe that it forms the basis for using the flat layer model and imparts a specific physical meaning to the coefficients involved in equations of this model.

From equation (23) one can obtain all of the particular cases considered earlier. For $\sigma_N = 0$ and $\gamma_N = 0$, we obtain the ideal MHD relationship (20). If, besides, $\partial\tilde{B}_3/\partial x^2 = 0$, i.e., $\tilde{\mu}_N = 0$, then the toroidal mode (13) for which $\bar{x}^1 = x_N^1(\omega)$ will be a solution of equation (23). If for the same axially symmetric oscillations it is assumed that $\sigma_N \neq 0$, then we obtain the equation given in a paper of Leonovich and Mazur (1987) which permits the transverse structure of toroidal modes to be investigated. In a subsequent discussion such eigen-modes unassociated with magnetosound will not be discussed. Both the dispersion ($\sigma_N \neq 0$) and damping at the ends ($\gamma_N \neq 0$) eliminate the property inherent in the solution (20). Which of the effects is more important depends on the relationship of appropriate parameters.

It is necessary to supplement equation (23) with boundary conditions with respect to the coordinate x^1. The particular formulation of these conditions depends on the asymptotic behaviour of solutions as one moves away from a resonance surface. In the opacity region the requirement of boundedness is a natural one, which leads to the condition for falling-down of the solution. In the non-opacity region one should specify the condition for energy escape from the resonance surface. This means that a source of Alfven waves is provided by magnetosound and there are no Alfven waves approaching the resonance surface from an asymptotically distant region.

An obvious feature of the solutions of equation (23), which is attributable to the smallness of the parameters ε and $\delta_\pm$, is their small-scale character, i.e., a typical scale of the solutions is much smaller than the inhomogeneity scale of the medium and, as we shall be assuming, than the typical scale of variation of the function $\tilde{\mu}_N$ in coordinate x^1. This allows us to write the solution as

$$\tilde{F}_N(x^1, x^2, \omega) = \tilde{\mu}_N(x^1, x^2, \omega)\tilde{Q}_N(x^1, \omega), \quad (24)$$

where $\tilde{Q}(x^1, \omega)$ is a function obeying the equation

$$\sigma_N(x^1)\frac{\partial^2 \tilde{Q}}{\partial x^{1^2}} + \left[\frac{(\omega + i\gamma_N(x^1))^2}{\Omega_N^2(x^1)} - 1\right]\tilde{Q}_N = 1, \quad (25)$$

and the same boundary conditions with respect to the variable x^1 as does $\tilde{F}_N$. The function $\tilde{Q}_N(x_1, \omega)$ does not depend on magnetosound field $\tilde{B}_3$ and is totally determined by the behaviour of magnetospheric parameters. It may be shown that it is an analytic function of a complex variable ω, whose singularities all lie in the half-plane $\mathrm{Im}\,\omega < 0$. Besides, it satisfies the relation

$$\tilde{Q}_N(x^1, -\omega) = \tilde{Q}_N^*(x^1, \omega). \quad (26)$$

These properties are the easiest to establish by expressing Q_N in terms of Green's function equation (25) and using the known properties of it. The property of (26) enables us to restrict ourselves to defining the function $\tilde{Q}_N$ for positive values of ω only.

The solution of equation (23) and the zeroth approximation expression (19) which are understood as generalized functions are close to each other. However, when considered to be ordinary functions of the coordinate x^1, they can differ greatly (at point x_N^1, for example). As far as the functions H_N and $\bar{H}_N$ are concerned, however, their difference h_N is small as compared with each of them in the usual sense. Therefore one may believe that the total spatial structure of a monochromatic Alfven wave in the main approximation is defined by the relation

$$\tilde{B}_2(x^1, x^2, x^3, \omega) = \sum_N \tilde{F}_N(x^1, x^2, \omega)H_N(x^1, x^3)$$
$$= \sum_N \tilde{\mu}_N(x^1, x^2, \omega)\tilde{Q}_N(x^1, \omega)H_N(x^1, x^3). \quad (27)$$

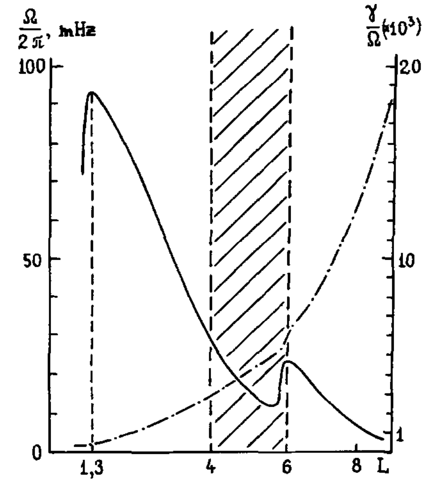

FIG. 2. PLOTS OF THE FUNCTIONS $\Omega(x^1)$ (SOLID LINE) AND $\gamma(x^1)$ (BROKEN LINE).
As the coordinate x^1, the McIllwain parameter L is chosen. The transition region, which is also called in the text the dissipative layer, is shaded; in this layer the Alfven wave energy is absorbed by magnetospheric plasma electrons.

The properties of the functions $\tilde{Q}_N(x^1, \omega)$ are determined by the behaviour of the coefficients of equation (25). Figures 2 and 3 show schematic plots of the functions $\Omega(x^1)$, $\gamma(x^1)/\Omega(x^1)$, $\rho(x^1)$ and $\alpha(x^1)$ for the dayside sector of the magnetosphere. They are constructed on the basis of empirical data for certain average conditions (Sergeev and Tsyganenko, 1980; Kamide and Matsushita, 1979). For a qualitative representation of the variation of the functions $\Omega_N(x^1)$, $\sigma_N(x^1)$ and $\gamma_N(x^1)$, one can use the WKB relationships $\Omega_N = N\Omega$, $\sigma_N = \sigma$, and $\gamma_N = \gamma$. The plot of the function $\Omega(x^1)$ has two typical features, namely a maximum on the shell $L \approx 1,3$ and a steep knee on $L = 4$–6 due to the plasmapause. The magnitude of $\sigma(x^1) = \rho^2(x^1)$ $e^{-i\alpha(x^1)}$ varies from real negative values in the inner

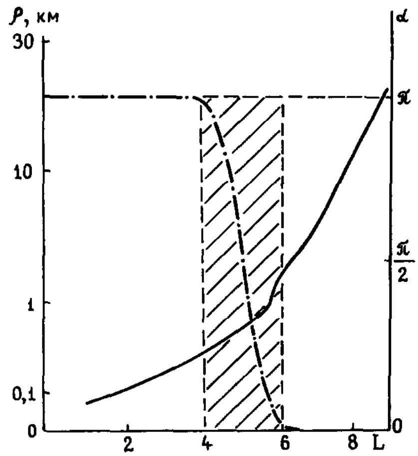

FIG. 3. PLOTS OF THE FUNCTIONS $\rho(x^1)$ (SOLID LINE) AND $\alpha(x^1)$ (BROKEN LINE).

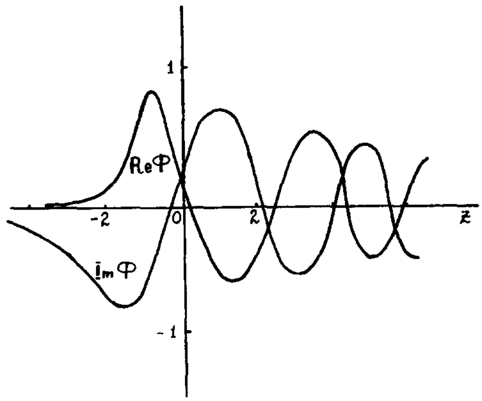

FIG. 4. PLOTS OF THE REAL AND IMAGINARY PARTS OF THE FUNCTION $\phi(z)$ DESCRIBING THE SPATIAL STRUCTURE OF A STANDING ALFVEN WAVE PERPENDICULAR TO THE MAGNETIC SHELLS NEAR A RESONANCE SHELL.

cold part of the magnetosphere to real positive values in the outer hot magnetosphere. In the dissipative region between them the values of σ are complex. The thickness of this region does not seem to be very large, $\Delta L \sim 1$–3, and the position coincides roughly with the plasmapause.

That the solutions of equation (25) are small-scale ones and are localized near the resonance surface permits these solutions to be determined explicitly. For that purpose, the expression in square brackets (25) should be expanded near the resonance surface. Two substantially different cases of the position of this surface are possible, namely in the region of monotonic variation of the function $\Omega(x^1)$ and near its extrema.

5. EXCITATION OF ALFVEN WAVES IN THE REGION OF MONOTONIC VARIATION OF THE FUNCTION $\Omega(x^1)$

Let $x^1 = \bar{x}^1$ be a certain fixed magnetic shell which is located sufficiently far from the extrema of the function $\Omega_N(x^1)$. Near it, one can put

$$\Omega_N(x^1) = \bar{\Omega}_N(1 - x/l_N), \quad x = x^1 - \bar{x}^1. \tag{28}$$

Here it is assumed that the function $\Omega_N(x^1)$ decreases with increasing x^1, as is the case in most of the magnetosphere. We assume first that the phase $\alpha_N = 0$, i.e., the quantity $\sigma_N = \rho_N^2$ is real and positive.

The solution of equation (25) is essentially different from zero in the vicinity of the shell $\bar{x}^1$ only if $\omega^2 \approx \bar{\Omega}_N^2$ i.e., if the resonance surface does not lie far from the shell $\bar{x}^1$. In this case equation (25) may be reduced to the form

$$\rho_N^2 \frac{\partial^2 \tilde{Q}_N}{\partial x^{1^2}} + \left[\frac{2x}{l_N} + \frac{(\omega + i\gamma_N)^2}{\bar{\Omega}_N^2} - 1\right]\tilde{Q}_N = 1.$$

Its solution, when $\omega > 0$ (i.e., actually when $\omega \approx \Omega_N$), can be respresented as

$$\tilde{Q}_N(x^1, \omega) = \left(\frac{l_N}{2\rho_N}\right)^{2/3} \phi\left[2\left(\frac{l_N}{2\rho_N}\right)^{2/3} \times \left(\frac{x}{l_N} + \frac{\omega - \bar{\Omega}_N + i\gamma_N}{\bar{\Omega}_N}\right)\right]. \tag{29}$$

Here $\phi(z)$ is a solution of the inhomogeneous Airy equation

$$\phi'' + z\phi = 1,$$

satisfying the condition of boundedness for $z \to -\infty$ and the runaway condition for $z \to +\infty$. The properties of the function $\phi(z)$ are described in the Appendix and the Re $\phi(z)$ and Im $\phi(z)$ plots for real values of z are presented in Fig. 4. It is possible, through simple transformations, to reduce the expression (29) to the form in which the dependence on an arbitrarily chosen surface $\bar{x}^1$ vanishes:

$$\tilde{Q}_N(x^1, \omega) = \left(\frac{l_N}{2\rho_N}\right)^{2/3} \phi\left[2\left(\frac{l_N}{2\rho_N}\right)^{2/3} \times \frac{\omega - \Omega_N(x^1) + i\gamma_N(x^1)}{\Omega_N(x^1)}\right]. \tag{30}$$

Here the quantity l_N should be understood thus

$$l_N = l_N(x^1) = \left|\frac{\mathrm{d} \ln \Omega_N(x^1)}{\mathrm{d}x^1}\right|^{-1}. \tag{31}$$

The expression (30) is especially convenient to use when it is necessary to regard $\tilde{Q}_N$ as a function of frequency ω.

The properties of the solution $\tilde{Q}_N(x^1, \omega)$ depend substantially on the relation between the growth rate γ_N and the value of $\tau_N = (l_N/2\rho_N)^{2/3}\Omega_N^{-1}$ which will be referred to as the dispersion time. If $\gamma_N \ll \tau_N^{-1}$, then the form of the functions (29) and (30) is determined mainly by the dispersion effect, i.e., by the presence of a term with the second-order derivative in the equation for $\tilde{Q}_N$. The argument of the function ϕ in this case is nearly real. Typical scales of the solution in the variables x^1 and ω are, respectively, $\Delta x_N = \rho_N^{2/3} l_N^{1/3}$ and $\Delta\omega_N = \tau_N^{-1} = (\rho_N/l_N)^{2/3}\Omega_N$ and the value of the solution at the maximum $\tilde{Q}_N \sim (l_N/\rho_N)^{2/3}$. As one moves from the resonance surface away into the transparence region, where the solution has the form of a runaway wave, the damping on the ionosphere comes into play. It leads to a weak exponential decrease of the amplitude. When $x \gg \Delta x_N$, we have

$$|\tilde{Q}_N| = \pi^{1/2}\left(\frac{l_N}{2\rho_N}\right)^{2/3}\left(\frac{x}{\Delta x_N}\right)^{-1/4} \times \exp\left[-2\gamma_N\tau_N(x/\Delta x_N)^{1/2}\right]. \quad (32)$$

If $\gamma_N \gg \tau_N^{-1}$, then the decisive role is played by dissipation. The form of the solution in this limit is most readily obtained by dropping in (25) the differential term. Then

$$\tilde{Q}_N(x^1, \omega) = \frac{1}{2}\left(\frac{x}{l_N} + \frac{\omega - \bar{\Omega}_N + i\gamma_N}{\bar{\Omega}_N}\right)^{-1} = \frac{1}{2}\frac{\Omega_N(x^1)}{\omega - \Omega_N(x^1) + i\gamma_N(x^1)}.$$

The typical scales of this solution are $\Delta x_N = (\gamma_N/\Omega_N) l_N$ and $\Delta\omega_N = \gamma_N$ and the value at the maximum is $\tilde{Q}_N \sim \Omega_N/\gamma_N$.

The inequality $\gamma_N \gg \tau_N^{-1}$ will be satisfied for all N if

$$\gamma\tau_d \ll 1, \quad (33)$$

with $\tau_d = (l/\rho)^{2/3}\Omega^{-1}$. Bearing in mind the definitions of (10) and (17) we see that the condition (33) is equivalent to the inequality $\delta \ll \varepsilon^{2/3}$. Estimates show that for typical conditions throughout most of the magnetosphere $\gamma\tau_d \sim \delta\varepsilon^{2/3} \sim 0.1$–$1$, i.e., it is more likely that the inequality (33), rather than an inverse inequality, is satisfied. It is quite clear that for $\gamma\tau_d \sim 1$ the role of dispersion and dissipation is equally important and none of the effects can be ignored.

Let, now, the resonance surface be located in the transition region which occupies in the magnetosphere the magnetic shells on which the thermal velocity of electrons v_e coincides, on the order of magnitude, with Alfven velocity A. In this region an effective Cherenkov damping of Alfven waves due to electrons is taking place; therefore, it will be referred to here as the dissipative layer. In this layer the quantity Λ^2 and, therefore, σ_N have a negative imaginary part. The function $\sigma_N(x^1)$, during the transition from the inner to the outer magnetosphere, varies in a complex plane from negative real values to positive values, without going to zero anywhere.

It is easy to to see that instead of (30) we have in this case

$$\tilde{Q}_N(x^1, \omega) = \left(\frac{l_N}{2\rho_N}\right)^{2/3} e^{i\alpha_N/3} \times \phi\left[2\,e^{i\alpha_N/3}\left(\frac{l_N}{2\rho_N}\right)^{2/3}\frac{\omega - \Omega_N(x^1) + i\gamma_N(x^1)}{\Omega_N(x^1)}\right]. \quad (34)$$

The presence of the terms $\exp(i\alpha_N/3)$ alters substantially the asymptotic of the function $\tilde{Q}_N$. Thus, for $\gamma_N\tau_N \ll 1$ we have

$$|\tilde{Q}_N| = \pi^{1/2}\left(\frac{x}{\Delta x_N}\right)^{-1/4}\left(\frac{l_N}{2\rho_N}\right)^{2/3}\exp\left[-\frac{2}{3}\left(\frac{x}{\Delta x_N}\right)^{3/2} \times \sin\frac{\alpha_N}{2} - 2\gamma_N\tau_N\left(\frac{x}{\Delta x_N}\right)^{1/2}\cos\frac{\alpha_N}{2}\right], \quad x \gg \Delta x_N.$$

If $\alpha_N = 0$, then we obtain the previous expression (32). But when $\alpha_N \gg (\gamma_N\tau_N)^3$, the decrease of the amplitude is determined by the parameter α_N, i.e., from the physical standpoint, by the Landau damping due to electrons. Deep in the dissipative layer the value $\alpha_N \sim 1$ and, hence, a typical length of the decrease is Δx_N, i.e., equals the wavelength. Thus, in the dissipative layer the Landau damping leads to a rapid absorption of the wave energy and virtually does not permit the wave to propagate from the resonance surface.

Alfven wave absorption in the dissipative layer bears, we believe, a direct relation to the phenomenon of red arcs (Slater *et al.*, 1987). According to Cornwall *et al.*'s (1971) theory, the heating of suprathermal electrons which are responsible for red airglow of atmospheric oxygen, is produced by the Cherenkov absorption of Alfven waves; hence, it can occur in the dissipative layer only.

6. ALFVEN WAVE EXCITATION NEAR EXTREMA OF THE FUNCTION $\Omega_N(x^1)$

The formulae given in previous sections are inapplicable when the resonance surface lies in the immediate vicinity of the extremum point of the function $\Omega_N(x^1)$. Although the appropriate regions occupy a small part of the magnetosphere, the study of Alfven waves in them is of interest from a fundamental standpoint, especially when the combination of the form of an extremum (maximum or minimum) and the sign of dispersion produces the possibility that the wave is locked in the transverse direction. In this case the Alfven wave field is bounded in the longitudinal direction by the ionospheric ends and in the transverse direction, in coordinate x^1, by wave reflection points. Such a phenomenon is natural for calling the Alfvenic resonator. A possibility of its existence was first pointed out by Gul'elmi and Polyakov (1983). The maximum of the function $\Omega_N(x^1)$ on the shell $L \approx 1,3$ gives an example of such a resonator. For the sake of definiteness we shall restrict our attention just to this case.

Near the maximum we put

$$\Omega_N(x^1) = \bar{\Omega}_N(1 - x^2/2a_N^2), \quad x = x^1 - \bar{x}_N^1,$$

where $\bar{x}_N^1$ is a coordinate of the maximum of the function $\Omega_N(x^1)$ [not to be confused with the resonance surface coordinate $x_N^1(\omega)$], and a_N is the inhomogeneity scale. Assuming $\sigma_N = -\rho_N^2$ and introducing the designations

$$\zeta = x/\mathscr{C}, \quad \mathscr{C} = (a_N \rho_N)^{1/2},$$

$$\lambda = (a_N/\rho_N)[1 - (\omega + i\gamma_N)^2/\bar{\Omega}_N^2],$$

we reduce equation (25) to the form

$$\frac{d^2\tilde{Q}_N}{d\xi^2} + (\lambda - \zeta^2)\tilde{Q}_N = \frac{a_N}{\rho_N}.$$

Let us introduce in our treatment the eigen-values and normalized eigen-functions of the appropriate homogeneous equation

$$\lambda_n = 2n+1,$$

$$y_n(\zeta) = \pi^{-1/4} 2^{-n/2} (n!)^{-1/2} e^{-\xi^2/2} H_n(\zeta),$$

where $n = 0, 1, 2, \ldots$ are integer non-negative numbers, and $H_n(\zeta)$ are Hermitian polynomials. Using Green's function

$$G(\zeta, \zeta', \lambda) = \sum_{n=0}^{\infty} \frac{y_n(\zeta) y_n(\zeta')}{\lambda - \lambda_n}$$

we find the solution for the inhomogeneous equation

$$\tilde{Q}_N = \frac{a_N}{\rho_N} \sum_{n=0}^{\infty} \frac{c_n}{\lambda - \lambda_n} y_n(\zeta),$$

where

$$c_n = \int_{-\infty}^{\infty} y_n(\zeta)\, d\zeta.$$

Returning to the variables x^1 and ω, we rewrite this result as

$$\tilde{Q}_N(x^1, \omega) = \sum_{n=0}^{\infty} \frac{c_n \bar{\Omega}_N^2 y_n(x/\mathscr{C})}{(\omega + i\gamma_N)^2 - \Omega_{Nn}^2}, \tag{35}$$

where $\Omega_{Nn} = \bar{\Omega}_N(1 - \lambda_n \rho_N/2a_N)$ are eigen-frequencies of the Alfvenic resonator.

Let us consider monochromatic oscillations with frequency $\omega = \omega_0$. Let there be $\gamma_N \ll \Delta\Omega_N$, where $\Delta\Omega_N = (\rho_N/a_N)\bar{\Omega}_N$ is splitting of the Alfvenic resonator frequencies. Then, if frequency ω_0 is close to one of the eigen-frequencies, so that $|\omega_0 - \Omega_{Nn}| \ll \Delta\Omega_N$, we have

$$\tilde{Q}_N(x^1, \omega_0) = \frac{c_n \bar{\Omega}_N^2}{\omega_0^2 - \Omega_{Nn}^2} y_n\left(\frac{x}{\mathscr{C}}\right),$$

i.e., the external source excites one eigen-mode of the resonator. The inequality $\gamma_N < \Delta\Omega_N$ will be satisfied for all N, provided that the condition

$$\gamma \bar{\tau}_d \ll 1 \tag{36}$$

is satisfied, where $\bar{\tau}_d = (a_N/\rho_N)\Omega_N^{-1}$ is a typical dispersion time of the resonator. It is equivalent to the inequality $\delta \ll \varepsilon$ if under a typical length l in the definition of (17) we understand now the scale a_N. The condition (36) is more stringent than (33). For typical parameters of the resonator on $L \approx 1,3$, we have $\gamma\bar{\tau}_d = 10$–10^2, i.e., an inverse inequality $\gamma\bar{\tau}_d \gg 1$ is satisfied. This is also valid for other extremum points of the function $\Omega(x^1)$. In other words, in a real magnetosphere the resonator's properties of extrema $\Omega(x^1)$ cannot manifest themselves.

7. INFLUENCE OF AN ALFVEN WAVE UPON MAGNETOSOUND

Let us return to equation (8b) for a magnetosound wave. Its right-hand side is different from zero only in narrow layers near resonance surfaces. Let us consider one such surface and let us restrict ourselves to the case where it lies in the region of monotonic variation of $\Omega(x^1)$, outside the dissipative layer. Simple estimations show that the influence of the right-hand side of (8b) implies a jump of the derivative $\partial\tilde{B}_3/\partial x^1$ during a transition through the resonance layer, while the function $\tilde{B}_3$ itself varies quite a little inside it. In order to calculate the jump of the derivative, we integrate (8b) over the coordinate x^1 in the interval $(x_N^1 - \Delta/2, x_N^1 + \Delta/2)$. We choose the length of the interval of integration Δ to be much larger than a typical scale of variation of the function $\tilde{Q}_N(x^1, \omega)$ but much smaller than the scale of variation of the equilibrium parameters as well as the function $\tilde{B}_3$. Then, by factoring, on the right-hand side of (8b), slowly varying terms at point x_N^1 outside the integral sign, the integral of the function $\tilde{Q}_N$ can be extended to the interval $(-\infty, \infty)$. On the other hand, on the left-hand side of (8b) one can put $\Delta \to 0$. As a result, we obtain

$$\left\{\frac{\partial \tilde{B}_3}{\partial x^1}\right\}_{x_N^1} = \left(\frac{\partial}{\partial x^3}\frac{g_1}{g_2}\right)_{x^1=x_N^1} H_N(x_N^1, x^3) \times \tilde{\mu}_N(x_N^1, x^2, \omega) \int_{-\infty}^{\infty} \tilde{Q}_N(x^1, \omega)\,\mathrm{d}x^1. \quad (37)$$

Here we have introduced the notation

$$\{f(x^1, x^3)\}_{x_N^1} = f(x_N^1+0, x^3) - f(x_N^1-0, x^3).$$

The integral on the right-hand side of (37) will be calculated in the Appendix. Taking also the definition of (31) into account we have

$$\left\{\frac{\partial \tilde{B}_3(x^1, x^2, x^3, \omega)}{\partial x^1}\right\}_{x_N^1} = -\mathrm{i}\frac{\pi}{2}\left[\frac{\partial}{\partial x^3}\frac{g_1(x_N^1, x^3)}{g_2(x_N^1, x^3)}\right] \times H_N(x_N^1, x^3)\left|\frac{\partial \ln \Omega_N(x_N^1)}{\partial x^1}\right|^{-1} \oint e(x_N^1, x^{3'}) \times \frac{\partial \tilde{B}_3(x_N^1, x^2, x^{3'}, \omega)}{\partial x^2}\,\mathrm{d}x^{3'}. \quad (38)$$

Let us stress once more that this condition does not have any analogue in the flat layer model with straight field-lines.

As a result, the problem for a magnetosound wave may be formulated thus. It is necessary to find the solution of equation (8b), with its right-hand side equal to zero, which satisfies:

(i) certain inhomogeneous boundary conditions on a certain outer boundary (the magnetopause, say) which plays the role of a wave source;
(ii) homogeneous boundary conditions on the ionosphere; and
(iii) the matching condition (38) on resonance magnetic surfaces.

The particular formulation of the boundary conditions on the external boundary and on the ionosphere as well as, generally, the solution of the problem of a magnetosound wave, must be the subject of a separate study. The purpose of this section has been to show that the problem for magnetosound is formulated independently of the problem for an Alfven wave.

8. CONCLUSIONS

Let us formulate the main results of this paper.

(1) Equations have been derived, which describe monochromatic MHD oscillations of an axisymmetric magnetosphere. The equations take account of the relationship between magnetosound and Alfven wave as well as the transverse dispersion of the latter one and its dissipation in the ionosphere. The Alfven wave dispersion is caused by the combined action of the effects of electron inertia and finite Larmor radius of the ions.

(2) Formulae have been obtained, which define the spatial structure of an Alfven wave excited by a monochromatic magnetosound. They solve the problem of Alfvenic resonance in an axially symmetric system. The formulae have been derived in terms of perturbation theory based on the presence of two dimensionless small parameters $\varepsilon \ll 1$ and $\delta \ll 1$ [see the definitions of (10) and (17)]. The parameter ε characterizes the transverse dispersion of an Alfven wave, and δ describes its dissipation in the ionosphere.

(3) According to the formulae obtained, the longitudinal structure of an Alfven wave is determined by the solution of a one-dimensional (in coordinate x^3) problem for eigen-values (11). The transverse coordinate x^1 is involved in it as a parameter on which the eigen-frequencies $\Omega_N(x^1)$ and the eigen-functions $H_N(x^1, x^3)$ depend. These latter are standing waves with N-nodes on the line of force. By specifying a model of the geomagnetic field geometry and plasma density, the problem (11) is easily solved numerically, while for the harmonics with $N \gg 1$ the solution is given by the formulae of WKB approximation (14).

(4) The transverse structure of an Alfven wave is determined by the solution of the one-dimensional equation (23), whose coefficients are integral characteristics of lines of force. From it we find that the Alfven wave is concentrated near resonance magnetic surfaces which are defined by the condition $\Omega_N(x^1) = \omega$, where ω is the magnetosound frequency. In the ideal MHD approximation the Alfven wave field has a peculiarity on the resonance surface. It is eliminated by effects of transverse dispersion of Alfven waves and of their dissipation on the ionospheric ends.

(5) These results explain why previous works were successful; their authors restricted attention to considering toroidal eigen-modes or utilized a simple model of a flat plasma layer. It is apparent that the longitudinal structure and the frequency of an Alfven wave excited by a magnetosound, coincide with those of a toroidal mode localized on a resonance surface. On the other hand, the transverse structure of the wave in an axially symmetric system, as in the flat-layer model, is described by the one-dimensional equation (23). Despite the intuitively clear character, these conclusions are not obvious ones and represent one of the main results of this paper. It is funda-

mentally important that, unlike the flat-layer model, the coefficients of equation (23) have a physically clear meaning and can be determined numerically.

(6) The transverse equation (23) has been solved for two different cases of the position of a resonance surface, namely in the region of monotonic variation of the function $\Omega_N(x^1)$ and near its extrema. The form of the solution depends substantially on the relative role of the dispersion and dissipation effects.

In the region of monotonic variation of the function $\Omega_N(x^1)$ when $\delta \ll \varepsilon^{2/3}$ the transverse dispersion effect is dominant. The Alfven wave has a maximum on the resonance surface and propagates from it perpendicularly to the magnetic shells, carrying along the energy which is drawn from the magnetosound. The wave amplitude decreases slowly with distance from the resonance surface as a consequence of the damping on ionospheric ends. If $\delta \gg \varepsilon^{2/3}$, then the properties of the solution are totally determined by the dissipation in the ionsophere. The wave field also has a maximum on the resonance surface, but the wave does not propagate and decreases monotonically on both sides of it. When $\delta \sim \varepsilon^{2/3}$ both effects are equally important for the field structure. In the Earth's magnetosphere typical values of $\delta\varepsilon^{-2/3} = 0.1$–$1$.

Near the extrema of the function $\Omega_N(x^1)$ the existence of an Alfvenic resonator is possible, in which the wave field is locked in the longitudinal direction by ionospheric ends and, in the transverse direction, by wave reflection points. Such a possibility is produced when at the minimum $\Omega_N(x^1)$ the dispersion is positive ($\Lambda^2 > 0$) or at the maximum is negative ($\Lambda^2 < 0$). The latter possibility occurs in the magnetosphere on the shell $L \approx 1,3$. Resonator properties of the extrema may manifest themselves if $\delta \ll \varepsilon$. Then the dissipation effect is weaker than the dispersion effect and the magnetosound can excite separate eigen-modes of the resonator. However, in the Earth's magnetosphere this phenomenon does not seem to be realized because typical values of $\delta\varepsilon^{-1} = 10$–$100$.

(7) Attention has been paid to the existence in the magnetosphere of a dissipative layer which separates the inner magnetosphere with "cold" transverse dispersion of Alfven waves from the outer magnetosphere with "hot" dispersion. In this layer $v_e \sim A$ and Alfven waves undergo a strong Landau damping due to electrons. If the resonance surface lies in the dissipative layer, then the driven Alfven wave is absorbed in its immediate vicinity on a scale of the order of the transverse wavelength.

(8) An analysis has been made of the back influence of an Alfven wave to magnetosound. It has been shown that it implies a jump of the derivative of the magnetosound wave field during the transition through the resonance surface. It has been found that, with the actually existing curvature of geomagnetic field-lines, the matching conditions on the resonance surface have a fundamentally different character from those in the flat-layer model. It has been shown that these conditions can be formulated in terms of the magnetosound field, which permits the problem of its propagation to be solved, irrespective of the problem of an Alfven wave.

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APPENDIX

One of the solutions of the inhomogeneous Airy equation

$$y''+zy = 1 \tag{A1}$$

is the function

$$\Psi(z) = \int_0^\infty \exp\left(-\frac{u^3}{3}-uz\right)\mathrm{d}u. \tag{A2}$$

Its asymptotics on Stokes lines $z = \xi\,\mathrm{e}^{\mathrm{i}2\pi n/3}$ (where $\xi > 0$) of the appropriate homogeneous equation have the form

$$\Psi(\xi) = 1/\xi,$$

$$\Psi(\xi\,\mathrm{e}^{\pm\mathrm{i}2\pi/3}) = \pi^{1/2}\xi^{-1/4}\exp\left[\mp\mathrm{i}\frac{2}{3}\xi^{3/2}\pm\mathrm{i}\frac{\pi}{12}\right].$$

The function $\Psi(z)$ is such a particular solution of the inhomogeneous equation which does not involve any running waves on Stokes line $z = \xi$. Consequently, in the sectors $0 < \arg z < 2\pi/3$ and $-2\pi/3 < \arg z < 0$ adjacent to it a falling exponent which is lost against the background of the term $1/z$ is able to appear in the asymptotic. This means that the asymptotic $\Psi(z) = 1/z$ is valid inside the entire sector $|\arg z| < 2\pi/3$. On the other two Stokes lines the falling exponents become running functions, in which case on the lower line the wave runs away into infinity and on the upper line the wave runs from infinity. During a transition through these lines there appears a growing exponent. For example, along the ray $z = -\xi$ we have

$$\Psi(-\xi) = \pi^{1/2}\xi^{-1/4}\exp\left(\tfrac{2}{3}\xi^{3/2}\right).$$

We also want to note the value in zero: $\Psi(0) = 3^{1/3}\Gamma(1/3)$. It is easy to see that, in addition to $\Psi(z)$, the functions $v\Psi(vz)$, if $v^3 = 1$, are also solutions of equation (A1). Apart from the initial case $v = 1$, there are also two different possibilities that $v = \exp(\pm\mathrm{i}2\pi/3)$. We are interested in the function

$$\phi(z) = \mathrm{e}^{-\mathrm{i}2\pi/3}\Psi(z\,\mathrm{e}^{-\mathrm{i}2\pi/3}).$$

Its properties immediately follow from the properties of $\Psi(z)$. In particular, the asymptotics on the real axis are

$$\phi(\xi) = -\pi^{1/2}\xi^{-1/4}\exp\left(\mathrm{i}\frac{2}{3}\xi^{3/2}+\mathrm{i}\frac{\pi}{4}\right),$$

$$\phi(-\xi) = 1/\xi.$$

It is evident from them that the function $\phi(\xi)$ satisfies the condition for run-away of the wave. For $\phi(z)$, one can obtain its own integral representation. To do so, we must note that integration in (A2) can be performed along any ray in the plane of complex u lying in the sector $|\arg u| \leqslant \pi/6$ because the term $\mathrm{e}^{-u^3/3}$ guarantees convergence of the integral in this sector. On making substitutions $z \to z\,\mathrm{e}^{-\mathrm{i}2\pi/3}$ and $u \to v\,\mathrm{e}^{\mathrm{i}\pi/6}$, we obtain

$$\phi(z) = -\mathrm{i}\int_0^\infty \exp\left(-\mathrm{i}\frac{v^3}{3}+\mathrm{i}vz\right)\mathrm{d}v. \tag{A3}$$

Note that the integral here can be taken along any ray in the sector $(\pi/3) \leqslant \arg v \leqslant 0$.

Next, we calculate two integrals which are needed. The first of them is

$$\mathscr{I}_1(\alpha,\varepsilon,\beta) = \int_{-\infty}^\infty \phi[\mathrm{e}^{\mathrm{i}\alpha/3}(\xi+\mathrm{i}\varepsilon)]\,\mathrm{e}^{-\mathrm{i}\beta\xi}\,\mathrm{d}\xi. \tag{A4}$$

The parameters α, ε and β are assumed real and $0 \leqslant \alpha \leqslant \pi$. Using the representation (A3) and using in it integration along the ray $v = t\,\mathrm{e}^{-\mathrm{i}\alpha/3}$ we have

$$\mathscr{I}_1 = -\mathrm{i}\int_{-\infty}^\infty \mathrm{d}\xi\int_0^\infty \mathrm{d}t\exp\left[-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha}\frac{t^3}{3}-\varepsilon t -\mathrm{i}\frac{\alpha}{3}+\mathrm{i}(t-\beta)\xi\right].$$

On changing the order of integration, we easily obtain

$$\mathscr{I}_1(\alpha,\varepsilon,\beta) = -\mathrm{i}2\pi\theta(\beta)\exp\left(-\varepsilon\beta-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha}\frac{\beta^3}{3}-\mathrm{i}\frac{\alpha}{3}\right). \tag{A5}$$

For $\alpha = 0$ and $\beta = 0$, a separate calculation yields

$$\mathscr{I}_1(0,\varepsilon,0) = -\mathrm{i}\pi. \tag{A6}$$

The second integral is

$$\mathscr{I}_2(\alpha,\varepsilon) = \int_{-\infty}^\infty |\phi(\mathrm{e}^{\mathrm{i}\alpha/3}(\xi+\mathrm{i}\varepsilon))|^2\,\mathrm{d}\xi. \tag{A7}$$

From (A3), we have

$$\mathscr{I}_2 = \int_{-\infty}^\infty \mathrm{d}\xi\int_0^\infty \mathrm{d}v\int_0^\infty \mathrm{d}v' \times\exp\left[\mathrm{i}\frac{v^3}{3}-\mathrm{i}\frac{v'^3}{3}-\mathrm{i}(v\,\mathrm{e}^{-\mathrm{i}\alpha/3}-v'\,\mathrm{e}^{\mathrm{i}\alpha/3})\xi -\varepsilon(v\,\mathrm{e}^{-\mathrm{i}\alpha/3}+v'\,\mathrm{e}^{\mathrm{i}\alpha/3})\right].$$

Next, let us integrate over rays $v = t\,\mathrm{e}^{\mathrm{i}\alpha/3}$ and $v' = t'\,\mathrm{e}^{-\mathrm{i}\alpha/3}$. On taking, first, the integral over ξ and, then, over t', we obtain

$$\mathscr{I}_2(\alpha,\varepsilon) = 2\pi\int_0^\infty \mathrm{d}t\exp\left(-\frac{2}{3}t^3\sin\alpha-2\varepsilon t\right) = \frac{2\pi}{(\sin\alpha)^{1/3}}\Psi\left(\frac{2\varepsilon}{(\sin\alpha)^{1/3}}\right). \tag{A8}$$