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AN ELECTROMAGNETIC FIELD, INDUCED IN THE IONOSPHERE AND ATMOSPHERE AND ON THE EARTH'S SURFACE BY LOW-FREQUENCY ALFVEN OSCILLATIONS OF THE MAGNETOSPHERE: GENERAL THEORY

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Abstract—Within the framework of a model of the near-terrestrial environment which represents adequately its vertical stratification, an analytical study is made of the structure of an electromagnetic field induced in the near-terrestrial layers by low-frequency Alfvén oscillations of the magnetosphere (whose frequency is much lower than the ionospheric eigenfrequencies). We begin this study by solving the problem for monochromatic oscillations which have a given horizontal wave vector, with an arbitrary orientation with respect to the meridian (the geomagnetic field being oblique). On the basis of this solution, using an inverse Fourier transform we derive general formulae to express the electromagnetic oscillation field on the Earth's surface in terms of the electromagnetic field of an Alfvén wave on the lower boundary of the magnetosphere. A boundary condition for Alfvén waves at the ionosphere–magnetosphere interface is obtained as an independent result. This condition carries all information on the ionosphere and on lower-lying layers, needed to solve problems of Alfvén oscillations of the magnetosphere.

1. INTRODUCTION

Penetration of hydromagnetic oscillations of the magnetosphere through the ionosphere and the atmosphere to the Earth's surface depends substantially on their frequency. In the hydromagnetic range (from milliHertz to several Hertz) the ionosphere features a set of eigenfrequencies which are due to its resonance properties. It is known that waveguide propagation of fast magnetosonic waves is possible in the ionospheric $F2$-layer (Greifinger and Greifinger, 1968; Greifinger, 1972), and waveguide propagation of a low-frequency whistler mode is possible in the E-layer (Sorokin and Fedorovich, 1982; Mazur, 1988), while a resonator exists for Alfvén oscillations in the topside ionosphere (Polyakov and Rapoport, 1981; Belyaev *et al.*, 1987). The eigenfrequencies of these (different in their physical nature) waveguides and the resonator are remarkably about the same, of order 1 Hz, i.e. coincide with the upper limit of the hydromagnetic range. Thus, two substantially different cases are possible: the frequency of the oscillations considered is either comparable with ionospheric eigenfrequencies or is much smaller than they are. We shall limit ourselves to the second, simpler case when resonance oscillations are not excited in the ionosphere. Note that a very important class—standing Alfvén waves in the magnetosphere—belongs to such low-frequency oscillations.

Papers of Hughes (1974) and Hughes and Southwood (1976a,b) have made an important contribution to the theory of penetration of low-frequency hydromagnetic oscillations to the Earth's surface. Under the assumption of a horizontal homogeneity of ground layers they constructed an analytic theory for simple particular cases, and the more general cases were investigated numerically. An analytical theory was developed for a vertical geomagnetic field as well as for an inclined field, but only for disturbances which do not depend on the azimuthal coordinate (in other words, the disturbance wave vector lies in the meridian plane). It was also assumed that the characteristic vertical wavelength is much larger than the ionospheric thickness such that this latter can be viewed as a thin film characterized by integral Hall and Pedersen conductivities. Numerical studies were made of more complex cases of disturbances dependent on the azimuthal coordinate. Those studies also addressed the penetration to the Earth of disturbances described in the magnetosphere by the theory of field line resonances (Chen and Hasegawa, 1974; Southwood, 1974). The magnetosphere is simulated by a flat layer of inhomogeneous plasma. The subsequent development of the theory was addressed in review papers by Southwood and Hughes (1983) and by Lyatsky and Maltsev (1983). A series of papers by Alperovich and Fedorov (1984a,b) should also be mentioned. The influence of the horizontal ionospheric inhomogeneity (for a vertical geomagnetic field) was investigated theoretically by Ellis and Southwood (1983) and Glassmeier (1983, 1984). They showed that a sufficiently large gradient of integral ionospheric conductivities characteristic for high latitudes has a substantial influence upon the character of penetration of an Alfvén wave through the ionosphere. In particular, the rotation angle of the ellipse of polarization of the wave can differ from 90° (Glassmeier, 1983, 1984) as is predicted by theory for a horizontally-homogeneous ionosphere (see, for example, Hughes and Southwood, 1976a).

In a number of our earlier papers (Leonovich and Mazur, 1989a,b, 1990) we have constructed an analytical theory to describe the space-time structure of the standing Alfvén wave field in an axisymmetrical model of the magnetosphere. Such a model provides a more adequate description of the real magnetosphere as compared with the usually used model of a plane plasma layer and leads to a richer picture of the phenomenon. Thus, it makes it possible to study the influence of both the transverse and longitudinal plasma inhomogeneity and of curvature of geomagnetic field lines. The theory developed in the papers cited above provides a full description of Alfvén oscillations directly in the magnetosphere, i.e. above a certain conventional boundary between the ionosphere and the magnetosphere. The question naturally arises as to the electromagnetic field induced by magnetospheric oscillations below this boundary—in the ground layers and in the Earth. It appears that the analytic theories developed in the cited papers by Hughes and Southwood are inapplicable in this case. This is because the Alfvén oscillations in the magnetosphere are extremely small-scale transversally—their wavelength across the magnetic field is much less than the typical thickness of conducting ionospheric layers. For an inclined geomagnetic field, this means that the vertical wavelength of the disturbance also is small compared with the thickness of the ionosphere, and this latter cannot be regarded as a thin film.

This paper develops an analytic theory to describe an electromagnetic field induced by low-frequency Alfvén oscillations of the magnetosphere in the ground layers, namely in the ionosphere and the atmosphere, and in the Earth. The theory is directed towards the solutions in the magnetosphere obtained in our previous papers or, in other words, extends them to the ground layers. A correct matching of the solutions in the axisymmetrical magnetosphere is made with those in the vertically-inhomogeneous ground medium; specifically, the question of the position of the conventional boundary between the ionosphere and the magnetosphere is ascertained. As an intermediate result, formulae are obtained which govern the penetration into the ground layers of separate disturbance Fourier-harmonics, i.e. monochromatic oscillations, having a given horizontal wave vector which is arbitrarily oriented with respect to the meridian. When developing the analytic theory, we are using a realistic model of the vertical stratification of the ground layers. A typical vertical behaviour of the model parameters is given in Fig. 1. The particular form of the functions presented in it can be taken from both the experiment and some standard models. It is also essential that an arbitrary inclination of the geomagnetic field to the Earth's surface is allowed for in this case.

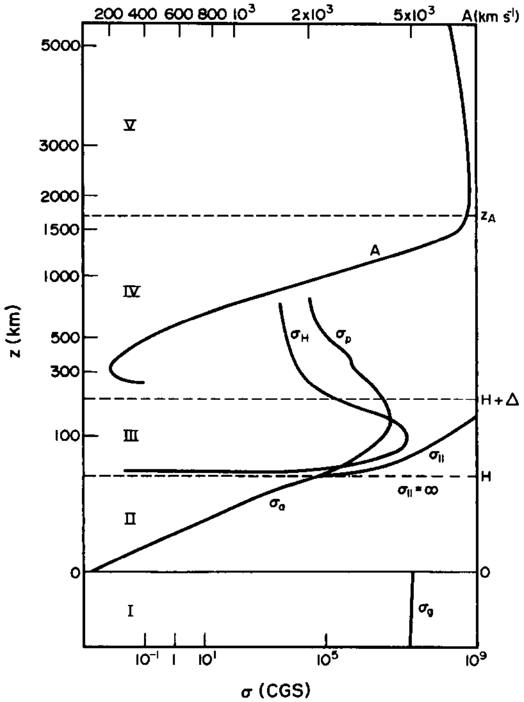

FIG. 1. TYPICAL HEIGHT PROFILES OF THE MAIN PARAMETERS OF THE MODEL OF THE MEDIUM.

Roman numerals denote the layers which have been described in detail in the text: I—Earth; II—atmosphere; III—lower ionosphere; IV—topside ionosphere; V—magnetosphere. The dash-dotted line represents the model value of longitudinal conductivity $\sigma_{\parallel} = \infty$.

The only important constraint which we impose on the model of ground layers is the assumption of their horizontal homogeneity. Incidentally, from simple physical considerations it is clear that such a homogeneity is required only at a length of the order of a typical horizontal scale of the oscillation field (~ 100 km). Horizontal inhomogeneities with a much larger scale can be readily introduced into the theory: for this purpose, it is necessary to use, in the formulae, local values of parameters of the medium. This reasoning allows us to consider in a self-consistent way the oscillations on a global scale, covering a significant part of the magnetosphere, the ionosphere, the atmosphere and of the Earth's surface. Of course, however, results of the theory are inapplicable in regions with large horizontal gradients of ionospheric conductivity such as, for example, electrojets or intensive precipitation regions.

In this paper it is believed that an Alfvén wave, rather than magneto-sound, is incident from the magnetosphere on the ground layers. At the same time, Kivelson and Southwood (1988) pointed out that, in the case of field line resonance, a magnetosonic wave, together with the Alfvén wave, must also be incident from the magnetosphere, and this is of great significance for the field pattern on the Earth. Indeed, in the magnetosphere model in the form of a "rectangular box", which was used by Kivelson and Southwood, the magnetosonic field, while propagating along a homogeneous magnetic field and a homogeneous (along it) plasma, reaches freely the ends which simulate the conjugate ionospheres. The situation is, however, quite different in the axisymmetrical model. In such a model, magneto-sound of the frequency range considered virtually does not reach the ionosphere because the inner part of the magnetosphere is an opacity region for it. The magnetosonic field amplitude there does not oscillate in space but decreases very rapidly in the earthward direction. This was briefly mentioned in a paper of Leonovich and Mazur (1989a); the Appendix to this paper gives a more detailed account of this issue.

Owing to its large size, this work is divided into two articles. In the first article, we have obtained general formulae to describe the space-time structure of the electromagnetic field in the ground layers for an arbitrary Alfvén wave in the magnetosphere. The derivation of the boundary condition for Alfvén waves on the above-mentioned conventional boundary between the ionosphere and the magnetosphere is given as an independent important result, and the question of the position of this boundary is addressed exhaustively. In the second article the obtained formulae are used for describing the field on the Earth induced by different types of Alfvén oscillations of the axisymmetric magnetosphere which were investigated in our earlier papers.

2. INPUT EQUATIONS AND A MODEL OF THE MEDIUM

Maxwell's equations for monochromatic oscillations with frequency ω will be written as

$$\operatorname{curl}\mathbf{E} = ik_0\mathbf{B}, \quad \operatorname{curl}\mathbf{B} = -ik_0\mathbf{E}+\frac{4\pi}{c}\mathbf{j}, \tag{1}$$

where $k_0 = \omega/c$, $\mathbf{E}$ and $\mathbf{B}$ are the disturbed electric and magnetic fields, respectively, and $\mathbf{j}$ is the conduction current. For the VLF oscillations of our interest here, the latter one has the form

$$\mathbf{j} = \sigma_{\parallel}\mathbf{E}_{\parallel}+\sigma_{\perp}\mathbf{E}_{\perp}+\sigma_{\mathrm{H}}\left[\frac{\mathbf{B}_0}{B_0}\mathbf{E}\right]. \tag{2}$$

Here $\mathbf{B}_0$ is the geomagnetic field, $\mathbf{E}_{\parallel}$ and $\mathbf{E}_{\perp}$ are the longitudinal (along $\mathbf{B}_0$) and transverse components of a disturbed electric field $\mathbf{E} = \mathbf{E}_{\parallel}+\mathbf{E}_{\perp}$, respectively, and $\sigma_{\perp}$, σ_{H} and $\sigma_{\parallel}$ are the components of the conductivity tensor. In gaseous media (atmosphere, ionosphere and magnetosphere), for oscillations with a frequency much lower than the gyrofrequency of ions, the latter ones are given by the formulae (see Ginzburg and Rukhadze, 1975)

$$\sigma_{\perp} = \frac{en_0c}{B_0}\left(\frac{S_e}{1+S_e^2}+\frac{S_i-i\omega/\omega_i}{1+S_i^2}\right), \tag{3a}$$

$$\sigma_{\mathrm{H}} = \frac{en_0c}{B_0}\left(\frac{1}{1+S_e^2}-\frac{1}{1+S_i^2}\right), \tag{3b}$$

$$\sigma_{\parallel} = \frac{en_0c}{B_0}\frac{1}{S_e}. \tag{3c}$$

Here e is the charge of a proton, n_0 is plasma density, $S_e = \nu_e/\omega_e$, $S_i = \nu_i/\omega_i$, where ν_e and ν_i are collision frequencies of electrons and ions, respectively, with neutral particles, and ω_e and ω_i are their gyrofrequencies. The parameters S_e and S_i characterizing the collisional ability of the respective plasma component, vary over a very broad range from values of S_e and $S_i \gg 1$ in the lower atmosphere to S_e and $S_i \ll 1$ in the topside ionosphere and magnetosphere. Note that at comparable temperatures of electrons and ions we have $\nu_e \ll \nu_i$.

When studying the electromagnetic field in the near-terrestrial region, the Earth's surface may, to quite an acceptable accuracy, be considered flat. Let us introduce a Cartesian system of coordinates, whose z-axis is directed along the vertical, the x-axis runs along the meridian from the equator to the pole, and the y-axis is directed along the parallel. The geomagnetic field will be considered homogeneous and directed at an angle χ to the z-axis (see Fig. 2). The main constraint which we impose on the model of the medium, i.e. the supposition about horizontal homogeneity, means that the conductivity tensor components depend only on the coordinate z. The spatial ground region is

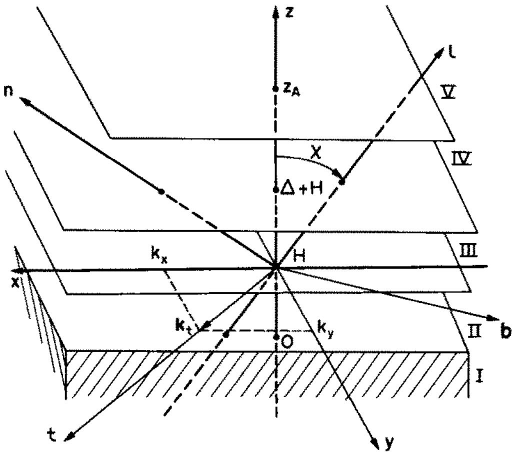

FIG. 2. THE MUTUAL POSITION OF THE THREE COORDINATE SYSTEMS USED IN THIS PAPER: (x, y, z), (t, b, z), (n, y, l).

divided into several layers. Let us describe them briefly (see also Fig. 1).

(I) The Earth ($z < 0$). The Earth has isotropic conductivity, which means $\sigma_{\parallel} = \sigma_{\perp} = \sigma_g$, $\sigma_H = 0$. A typical value of conductivity is rather large: $\sigma_g = (10^8 - 10^{10})\ \mathrm{s}^{-1}$ such that most of the subsequent results remain unchanged when passing to the limit of an ideally conducting Earth: $\sigma_g \to \infty$. The vertical dependence $\sigma_g(z)$ has, as a consequence of the multi-layered structure of the Earth's crust and mantle, a very complex character, but we shall confine ourselves to the simplest assumption $\sigma_g = \mathrm{const}$ which permits us to take the non-ideal terrestrial conductivity crudely into account.

(II) The atmosphere ($0 < z < H$). The conductivity of this layer (and of all the above-lying ones) is described by formulae (3). The atmosphere provides a strongly collisional medium where $S_e \gg 1$ and $S_i \gg 1$. The relationships (3) in this case yield $\sigma_{\parallel} = \sigma_{\perp} = \sigma_a$, $\sigma_H = \sigma_a/S_e$, where $\sigma_a = e^2 n_0/m_e \nu_e$. We shall assume that $\sigma_H = 0$, i.e. we shall consider the atmosphere also to be an isotropic medium. Atmospheric conductivity grows upwards very rapidly, approximately exponentially with a typical scale of about 5–10 km. In the lower part of the atmosphere, for the frequencies of our interest here, the inequality $\sigma_a \ll \omega/4\pi$ is satisfied with a large margin. The upper boundary of the atmosphere is defined by the condition $S_e \sim 1$, which gives $H = 80$–100 km. Near this boundary $\sigma_a = 10^5$–$10^6\ \mathrm{s}^{-1}$ and, hence, the inequality $\sigma_a \gg \omega/4\pi$ is fulfilled with a large margin. Note that the lowest ionospheric layer (D-layer) is included in the atmosphere here because the conductivity there is still isotropic.

(III) The lower ionosphere ($H < z < H+\Delta$). The collisional parameters there run through values equal to unity, at first S_e at 90–100 km altitude and then S_i at 120–140 km altitude. In this layer we shall assume $\sigma_{\perp} = \sigma_P$, where σ_P is the Pedersen conductivity, the expression for which is given by formula (3a), in which one should put $\omega/\omega_i = 0$. A maximum of Pedersen conductivity lies in a region where $S_i \sim 1$. Hall conductivity is concentrated in a layer where the inequalities $S_e < 1$ and $S_i > 1$ are satisfied. It may be identified with the morphologically distinguished E-layer. Typical maximum conductivity values of the dayside ionosphere are $\sigma_H = 3 \cdot 10^6\ \mathrm{s}^{-1}$ and $\sigma_P = 10^6\ \mathrm{s}^{-1}$. The thickness of the lower ionosphere, i.e. of a layer in which the Hall and Pedersen conductivities are concentrated, is $\Delta \sim 50$–100 km. The longitudinal conductivity $\sigma_{\parallel} \sim \sigma_P/S_e$ is much larger than σ_P and σ_H, and we shall assume that $\sigma_{\parallel} = \infty$. Thus, the ionosphere is a medium with sharply anisotropic conductivity.

(IV) The topside ionosphere ($H+\Delta < z < z_A$). In this case the plasma is nearly a collisionless one: $S_e \ll 1$ and $S_i \ll 1$. The longitudinal conductivity is very large such that one is still more justified in assuming that $\sigma_{\parallel} = \infty$. When $\nu_i \ll \omega$, the decisive role in (3a) is played by the term with frequency. In this case

$$\sigma_{\perp} = -\mathrm{i}\frac{\omega}{4\pi}\frac{c^2}{A^2} = -\mathrm{i}\frac{c}{4\pi}\frac{k_A^2}{k_0}, \tag{4}$$

where it is designated: $A = B_0/\sqrt{4\pi n_0 m_i}$ is the Alfvén velocity, and $k_A = \omega/A$. In this layer $\sigma_H \ll |\sigma_{\perp}|$ and it will be assumed that $\sigma_H = 0$. The Alfvén velocity involved in (4) varies in the layer under consideration in a very wide range; it has a minimum value of 200–300 km s^{-1} in the F-layer and, while increasing rapidly, reaches a maximum of the order of 10^4 km s^{-1} at height $z_A = (1-2) \cdot 10^3$ km.

(V) The magnetosphere ($z > z_A$). The region above the Alfvén velocity maximum will be considered to be the magnetosphere proper. It differs from the topside ionosphere only in as much as the Alfvén velocity there varies slowly, i.e. a typical scale of its variation is of the order of the field line length.

Let us make several remarks on the boundaries between the different layers. There actually exists only one sharp boundary, the Earth's surface, where conductivity experiences a large jump. This is included in the model described here. Under real conditions there are no other sharp boundaries, and the layers insensibly shade into one another. In accordance with this, the parameters of the medium between the layers in our model change continuously; but there is one exception which we were unable to avoid. The model assumes that there is still another sharp boundary separating the atmosphere and the ionosphere, on which the longitudinal conductivity changes abruptly from a finite value to the value of $\sigma_\parallel = \infty$. We believe that this limitation of the model is not a large deviation from the reality. Indeed, the values of the functions $\sigma_\parallel(z)$ and $\sigma_\perp(z)$, starting from a certain height, differ in a very fast increasing manner and, already at 15–20 km from that, the value of $\sigma_\parallel$ exceeds $\sigma_\perp$ by several orders of magnitude. Moreover, it appears that the crucial formulae to be obtained below do not include explicitly the position of this boundary.

A condition for matching the solutions on sharp boundaries is provided by the requirement of continuity for the tangential components of the electric and magnetic oscillation fields. Matching on smooth transitions between the layers is accomplished through a join of solutions in the overlapping region, where solutions from the two neighbouring layers are applicable.

Components of disturbed electric and magnetic fields, the B_x-component, say, are functions of coordinates and time: $B_x = B_x(x, y, z, t)$. In virtue of stationarity of the medium, assumed here, the equations are simplified considerably after making a Fourier-transform in time

$$B_x(x, y, z, t) = \int_{-\infty}^{\infty} \tilde{B}_x(x, y, z, \omega)\, e^{-i\omega t}\, d\omega.$$

Equations (1) are written for separate Fourier-harmonics. In view of the horizontal homogeneity of the medium, it is also useful to carry out a Fourier-transform in coordinates x and y:

$$\tilde{B}_x(x, y, z, \omega) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \times \bar{B}_x(k_x, k_y, z, \omega)\, e^{ik_x x + ik_y y}.$$

We shall be using such Fourier-harmonics throughout most of the present paper and, for brevity, we shall generally not write out the dependence on the arguments k_x, k_y and ω. From the results reported in our previous papers (Leonovich and Mazur, 1989a,b, 1990) it follows that a typical transverse scale of magnetospheric solutions projected onto the ionosphere is about 1–100 km. This means that a typical value of the horizontal wave vector is $k_t = \sqrt{k_x^2 + k_y^2} \sim (1\text{–}10^{-2})\ \text{km}^{-1}$.

When considering a single spatial Fourier-harmonic, it is convenient to use a new coordinate system (t, b, z) rotated with respect to the first one around the z-axis (see Fig. 2). Let us introduce a two-dimensional wave vector $\mathbf{k}_t = (k_x, k_y)$ and let the t-axis be directed along it and the b-axis be directed in the horizontal plane perpendicularly to it. We have the relationships

$$\bar{B}_t = (k_x/k_t)\bar{B}_x + (k_y/k_t)\bar{B}_y,$$
$$\bar{B}_b = -(k_y/k_t)\bar{B}_x + (k_x/k_t)\bar{B}_y$$

and similarly for any other vector.

In the ionosphere and magnetosphere we shall be using the third coordinate system (n, y, l) rotated with respect to the first one in the meridional plane. They have a common y-axis, and the l-axis is directed along the geomagnetic field, while the n-axis is perpendicular to both of them. In this coordinate system

$$\bar{B}_n = \bar{B}_x \cos\chi + \bar{B}_z \sin\chi, \quad \bar{B}_\parallel = -\bar{B}_x \sin\chi + \bar{B}_z \cos\chi.$$

Here $\bar{B}_\parallel$ denotes vector $\mathbf{B}$ projected onto the l-axis (the longitudinal component of the vector).

To conclude this section, we shall write out, component-wise for a separate Fourier-harmonic, equations (1) in isotropic media, where $\sigma_\parallel = \sigma_\perp = \sigma$, $\sigma_H = 0$ and, consequently, $\mathbf{j} = \sigma\mathbf{E}$. These equations have the simplest form in the coordinate system (t, b, z). Let us introduce the designation

$$\kappa = 4\pi\sigma/c.$$

This quantity has the dimensions of a wave vector. From (1) follow the relationships

$$\bar{E}_t = -\frac{1}{\kappa - ik_0}\frac{\partial \bar{B}_b}{\partial z}, \quad \bar{E}_z = i\frac{k_t}{\kappa - ik_0}\bar{B}_b,$$
$$\bar{B}_t = \frac{i}{k_0}\frac{\partial \bar{E}_b}{\partial z}, \quad \bar{B}_z = \frac{k_t}{k_0}\bar{E}_b, \tag{5}$$

which express in terms of B_b and E_b the other components of fields and the equations for the first ones

$$\frac{\partial^2 \bar{E}_b}{\partial z^2} - (k_t^2 - ik_0\kappa)\bar{E}_b = 0, \tag{6}$$

$$\frac{\partial^2 \bar{B}_b}{\partial z^2} - \frac{\kappa'}{\kappa - ik_0}\frac{\partial \bar{B}_b}{\partial z} - (k_t^2 - ik_0\kappa)\bar{B}_b = 0. \tag{7}$$

3. THE ELECTROMAGNETIC FIELD IN THE EARTH

Let us introduce the designation $\kappa_g = 4\pi\sigma_g/c$. By taking account of the terrestrial homogeneity proposed here, we reduce equations (6) and (7) to the form

$$\bar{E}_b'' - k_g^2\bar{E}_b = 0, \quad \bar{B}_b'' - k_g^2\bar{B}_b = 0,$$

where it is designated

$$k_g = \sqrt{k_t^2 - ik_0\kappa_g}$$

and the radical sign is chosen such that $Re\, k_g > 0$. In limiting cases we have

$$k_g = \begin{cases} k_t, & k_t^2 \gg k_0 \kappa_g, \\ j(k_0 \kappa_g)^{1/2}, & k_t^2 \ll k_0 \kappa_g, \end{cases}$$

where $j = e^{-i\pi/4} = (1-i)/\sqrt{2}$. Bearing in mind the typical limits of variations of the parameters $k_t = 1$–$10^{-2}\,\text{km}^{-1}$, $\omega = 10^{-2}$–$10^{-1}\,\text{s}^{-1}$, and $\sigma_g = 10^8$–$10^{10}\,\text{s}^{-1}$, it is easy to see that both limiting cases are realizable. The typical range of possible values of $|k_g| = 1$–$10^{-2}\,\text{km}^{-1}$.

The solutions of equations (6) and (7), which are bounded when $z < 0$, have the form

$$\bar{E}_b(z) = \bar{E}_b(0)\,e^{k_g z}, \quad \bar{B}_b(z) = \bar{B}_b(0)\,e^{k_g z}. \tag{8a}$$

From this and from the equalities (5), we have

$$\bar{B}_t(z) = \bar{B}_t(0)\,e^{k_g z}, \quad \bar{E}_t(z) = \bar{E}_t(0)\,e^{k_g z}, \tag{8b}$$

with

$$\bar{B}_t(0) = i\frac{k_g}{k_0}\bar{E}_b(0), \quad \bar{E}_t(0) = -\frac{k_g}{\kappa_g}\bar{B}_b(0). \tag{9}$$

These last relationships will be used as the boundary conditions for the solution in the atmosphere.

Thus, the oscillation field decreases exponentially deep into the Earth with a typical scale $l_g = (Re\, k_g)^{-1}$. If $k_0\kappa_g$ is larger than or comparable with k_t^2, this decrease is attributable to spatial oscillations as is the case in the classical problem of the skin layer.

Using the WKB approximation it is easy to obtain a solution for the case of a weakly inhomogeneous Earth, whose inhomogeneity scale is much larger than l_g. In this case the results remain almost unaltered because the solution is concentrated near the terrestrial surface on a scale l_g where conductivity is assumed to change little.

In the case of a very high conductivity of the Earth, when $k_0\kappa_g \gg k_t^2$, from (9) follows

$$\bar{E}_b(0) = j\left(\frac{k_0}{\kappa_g}\right)^{1/2}\bar{B}_t(0), \quad \bar{E}_t(0) = -j\left(\frac{k_0}{\kappa_g}\right)^{1/2}\bar{B}_b(0),$$

and in the limit $\kappa_g \to \infty$, as should be the case, $\bar{E}_b(0) = \bar{E}_t(0) = 0$.

4. THE ELECTROMAGNETIC FIELD IN THE ATMOSPHERE

Atmospheric conductivity is much smaller than the Earth's conductivity and the inequality $k_t^2 \gg k_0\kappa_a$ is valid throughout the atmospheric height. Therefore, the term $ik_0\kappa_a$ in equation (6) can be regarded as a disturbance. In the main approximation we have

$$\bar{E}_b'' - k_t^2 \bar{E}_b = 0.$$

This equation, together with the equality (5) and the boundary condition (9), leads to the following solution:

$$\bar{B}_t(z) = \bar{B}_t(0)\left(\cosh k_t z + \frac{k_t}{k_g}\sinh k_t z\right),$$

$$\bar{E}_b(z) = \bar{E}_b(0)\left(\cosh k_t z + \frac{k_g}{k_t}\sinh k_t z\right), \tag{10}$$

with

$$\bar{E}_b(0) = -i(k_0/k_g)\bar{B}_t(0).$$

From this, in particular, follows

$$\bar{B}_t(0) = \bar{B}_t(H)\left(\cosh k_t H + \frac{k_t}{k_g}\sinh k_t H\right)^{-1} \tag{11}$$

and

$$\bar{B}_t(H) = i\frac{\lambda k_t}{k_0}\bar{E}_b(H), \tag{12}$$

where it is designated

$$\lambda = \frac{1 + (k_t/k_g)\tanh k_t H}{\tanh k_t H + k_t/k_g}.$$

The equality (12) will be used subsequently as the boundary condition for a solution in the ionosphere.

In the next order of perturbation theory one may take account of the term $ik_0\kappa$ in equation (6) and obtain corrections for the solution (10) and the relationship (12). Small corrections to the functions $\bar{B}_t$ and $\bar{E}_b$ are of no practical concern. The correction to the relationship (12) will, ultimately, transform into a corresponding correction to the boundary condition for an Alfvén wave at the upper boundary of the ionosphere and will define that part of the damping decrement, which is connected with dissipation in the atmosphere. This part consists of only a negligible portion of the total decrement, which is determined mainly by the dissipation in the ionospheric Pedersen layer. Therefore, in equation (6) one can totally neglect the atmospheric conductivity effect.

Quite a different situation applies to equation (7). The term $ik_0\lambda_a$ in the last bracket of this equation can also be neglected, but the factor $\kappa_a'/(\kappa_a - ik_0)$, caused by the inhomogeneity of atmospheric conductivity, plays the decisive role. Let us designate

$$S = \frac{1}{2}\frac{\kappa_a'}{\kappa_a - ik_0}$$

and let us rewrite equation (7) as

$$\bar{B}_b'' - 2S\bar{B}_b' - k_t^2\bar{B}_b = 0.$$

A typical scale of variation of the function $S(z)$ is of the order of the atmospheric height H. Assuming that the scale of variation of the solution is much smaller, we shall seek it using the WKB method. Setting $\bar{B}_b = \exp \Psi(z)$, we get

$$\Psi'^2 + \Psi'' - 2S\Psi' - k_t^2 = 0.$$

We seek the function Ψ by the successive approximation method: $\Psi = \Psi_0 + \Psi_1 + \cdots$. In the main order

$$\Psi_0'^2 - 2S\Psi_0' - k_t^2 = 0.$$

This quadratic equation has two roots:

$$\Psi_0' \equiv q_{1,2} = S \pm \sqrt{S^2 + k_t^2}.$$

In the next order, for each of them, we obtain the equation

$$2(q - S)\Psi_1' + q' = 0,$$

from which it follows that

$$\Psi_1 = -\tfrac{1}{2}\ln(q^2 + k_t^2).$$

On the basis of these results it is easy to write a general solution for $\bar{B}_b(z)$. It depends on two arbitrary constants. Using relationship (5), this solution can yield the expression for $\bar{E}_t(z)$. After that, the boundary condition (9) makes it possible to interconnect the arbitrary constants and to express them in terms of a single one. The result can be represented as

$$\bar{E}_t(z) = C\frac{k_0}{k_0 + i\kappa_a}\left[\frac{q_1}{\sqrt{2(q_1^2 + k_t^2)}}\exp\int_0^z q_1\,dz' + \frac{q_2}{\sqrt{2(q_2^2 + k_t^2)}}\exp\int_0^z q_2\,dz'\right], \quad (13a)$$

$$\bar{B}_b(z) = iC\frac{k_0}{k_t}\left[\frac{k_t}{\sqrt{2(q_1^2 + k_t^2)}}\exp\int_0^z q_1\,dz' + \frac{k_t}{\sqrt{2(q_2^2 + k_t^2)}}\exp\int_0^z q_2\,dz'\right]. \quad (13b)$$

When deriving expressions (13), we have neglected the extremely small value of $k_0 k_y / k_t \kappa_g$ as compared with unity.

In the lower part of the atmosphere, where $\kappa_a \ll k_0$, the value of S is very small and $q_{1,2} = \pm k_t$. From (13) it then follows

$$\bar{E}_t(z) = C \sinh k_t z, \quad \bar{B}_b(z) = i(k_0/k_t)C \cosh k_t z. \quad (14)$$

In the upper part of the atmosphere the inequality $\kappa_a \gg k_0$ is satisfied (with a very large margin). Therefore $S = 1/2h$, where

$$h(z) = \kappa_a(z)/\kappa_a'(z) \equiv \sigma_a(z)/\sigma_a'(z)$$

is a typical scale of variation of conductivity. As has already been stated, when describing the model of the medium, if it is sufficiently small, $h = 5$–10 km (and we shall assume that $k_t h \ll 1$) then

$$q_1 = \frac{1}{h} + k_t^2 h, \quad q_2 = -k_t^2 h.$$

Note also that

$$Q_{1,2} \equiv \int_0^H q_{1,2}\,dz = \int_0^H S\,dz \pm \int_0^H \sqrt{S^2 + k_t^2}\,dz = \tfrac{1}{2}\ln\frac{\kappa_a(H)}{k_0} + i\frac{\pi}{4} \pm Q,$$

where it is designated

$$Q = \int_0^H \sqrt{S^2 + k_t^2}\,dz.$$

Since $Re\,Q \gg 1$, near the upper boundary of the atmosphere, only the first terms can be retained in (13). Then

$$\bar{E}_t(z) = \bar{E}_t(H)\exp\left(-k_t^2\int_z^H h\,dz'\right), \quad (15a)$$

$$\bar{B}_b(z) = \bar{B}_b(H)\frac{\kappa_a(z)}{\kappa_a(H)}\frac{h(z)}{h(H)}\exp\left(-k_t^2\int_z^H h\,dz'\right), \quad (15b)$$

and

$$\bar{E}_t(H) = C[k_0/2\kappa_a(H)]^{1/2}\,e^{Q - i\pi/4},$$
$$\bar{B}_b(H) = -C[k_0\kappa_a(H)/2]^{1/2}h(H)\,e^{Q - i\pi/4}. \quad (16)$$

From relationships (16) it follows that

$$\bar{B}_b(H) = -\kappa_a(H)h(H)\bar{E}_t(H).$$

The coefficient $\kappa_a(H)h(H)$ involved here is of the same order of magnitude as the atmospheric conductivity-induced correction to the coefficient $i\lambda k_t/k_0$ in equality (12). For the same reason, as in the case of neglecting this correction, we must also neglect the coefficient $\kappa_a h$. Thus, in the approximation we have adopted

$$\bar{B}_b(H) = 0. \quad (17)$$

In order to avoid a misunderstanding, we wish to note that the numerical value of the coefficient $\kappa_a(H)h(H) \sim 10^3$ seems to be large; but, as follows from (12), the representative value of the coefficient of proportionality between the electric and magnetic field $k_t/k_0 \sim 10^5$–10^7, i.e. by several orders of magnitude larger.

The equality (17) will be used subsequently as the second boundary condition for the solution in the ionosphere. As far as the atmospheric solution is concerned, from this it follows that $\bar{B}_b(z) = 0$ throughout the atmosphere. The electric field is given by formula (13a) from which the limiting cases (14) and (15a) follow. In the order of magnitude $\bar{E}_t(H) \sim C$.

5. EQUATIONS FOR THE ELECTROMAGNETIC FIELD IN THE IONOSPHERE

In the ionosphere and magnetosphere equations (1) have the simplest form in the coordinate system (n, y, l). In view of the ideal longitudinal plasma conductivity, $\sigma_\parallel = \infty$, from them we obtain the relationships

$$\bar{E}_\parallel = 0, \quad \bar{B}_\parallel = -\bar{B}_n \tan\chi - \frac{k_y}{k_0}\bar{E}_n + \frac{k_x}{k_0\cos\chi}\bar{E}_y. \tag{18}$$

Let the other components of disturbed fields be reduced to a four-dimensional column vector

$$\varepsilon = \begin{pmatrix} \bar{E}_n \\ \bar{E}_y \\ \bar{B}_y \\ \bar{B}_n \end{pmatrix}$$

and let the equations for them be represented in matrix form

$$-\mathrm{i}\frac{\partial\varepsilon}{\partial z} = \hat{Q}\varepsilon + \hat{q}\varepsilon. \tag{19}$$

Matrices $\hat{Q}$ and $\hat{q}$ have the form

$$\hat{Q} = \begin{pmatrix} k_x\tan\chi & 0 & \dfrac{k_0}{\cos\chi} & 0 \\ 0 & k_x\tan\chi & 0 & -\dfrac{k_0}{\cos\chi} \\ -\dfrac{k_y^2}{k_0\cos\chi} & \dfrac{k_xk_y}{k_0\cos^2\chi} & k_x\tan\chi & -k_y\dfrac{\tan\chi}{\cos\chi} \\ -\dfrac{k_xk_y}{k_0} & \dfrac{k_x^2}{k_0\cos\chi} & -k_y\sin\chi & -k_x\tan\chi \end{pmatrix},$$

$$\hat{q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{\kappa_\perp}{\cos\chi} & -\dfrac{\kappa_H}{\cos\chi} & 0 & 0 \\ -\kappa_H\cos\chi & -\kappa_\perp\cos\chi & 0 & 0 \end{pmatrix}. \tag{20}$$

Here it is designated $\kappa_\perp = 4\pi\sigma_\perp/c$ and $\kappa_H = 4\pi\sigma_H/c$. In the lower ionosphere $\sigma_\perp = \sigma_P$ and it will be assumed that $\kappa_\perp = \kappa_P \equiv 4\pi\sigma_P/c$. In the topside ionosphere and the magnetosphere $\kappa_H = 0$ and, according to (4),

$$\kappa_\perp = -\mathrm{i}k_A^2/k_0. \tag{21}$$

The system of equations (19) is simplified substantially if we use in the space of four vectors a basis of vectors which are solutions of the auxiliary equation

$$-\mathrm{i}\frac{\partial\varepsilon}{\partial z} = \hat{Q}\varepsilon. \tag{22}$$

Since matrix $\hat{Q}$ does not depend on z, solutions of these equations can be sought in the form

$$\varepsilon(z) = \psi\, e^{\mathrm{i}k_z z}.$$

For vector ψ we obtain the eigenvalue problem

$$\hat{Q}\psi = k_z\psi.$$

It has the following solutions

$$k_z^{(1)} = k_z^{(2)} = k_x\tan\chi, \quad k_z^{(3)} = \mathrm{i}k_t, \quad k_z^{(4)} = -\mathrm{i}k_t.$$

They correspond to the Alfvén and magnetosonic waves in the limit $\omega = 0$. Indeed, for the first pair we have $k_\parallel = -k_x\sin\chi + k_z\cos\chi = 0$, and for the second pair we get $k^2 \equiv k_x^2 + k_y^2 + k_z^2 = 0$.

The given eigenvectors ψ^3 and ψ^4 correspond to the roots $k_z^{(3)}$ and $k_z^{(4)}$. Only one eigenvector ψ^1 corresponds to a doubly-degenerate root $k_z^{(1)} = k_z^{(2)}$. The second linearly independent solution, corresponding to the roots $k_z^{(1,2)}$, should be sought in the form

$$\varepsilon(z) = (\alpha z + \beta)\exp(\mathrm{i}k_z^{(1)}z).$$

On substituting into (22), we obtain for vectors α and β:

$$\hat{Q}\alpha = k_z^{(1)}\alpha, \quad \hat{Q}\beta = k_z^{(1)}\beta - \mathrm{i}\alpha.$$

Hence it is evident that one may put $\alpha = \mathrm{i}S\psi^1$ where S is some constant (arbitrary, in principle), whose value will be chosen later. Let us designate further $\beta = \psi^2$. Thus we have

$$\hat{Q}\psi^1 = k_z^{(1)}\psi^1, \quad \hat{Q}\psi^2 = k_z^{(1)}\psi^2 + S\psi^1,$$
$$\hat{Q}\psi^3 = k_z^{(3)}\psi^3, \quad \hat{Q}\psi^4 = k_z^{(4)}\psi^4. \tag{23}$$

These relationships can be written thus:

$$\hat{Q}\psi^n = \Lambda_m^n\psi^m, \tag{24}$$

which implies the summation over the recurring index. Numbers Λ_m^n compose a certain matrix $\hat{\Lambda}$. Let us also introduce matrix $\hat{\psi}$, whose columns are column vectors ψ^1, ψ^2, ψ^3 and ψ^4. The relationship (24) can then be represented as

$$\hat{Q}\hat{\psi} = \hat{\psi}\hat{\Lambda}.$$

Since vectors ψ^n are linearly independent, matrix $\hat{\psi}$ is a non-degenerate one, and from the last equality it follows that

$$\hat{\Lambda} = \hat{\psi}^{-1}\hat{Q}\hat{\psi}.$$

Column vectors ψ^n form in the 4-space a complete system and any vector $\varepsilon(z)$ that depends on z, can be expanded with respect to them, with the expansion coefficients being functions of z:

$$\varepsilon(z) = \psi^n F_n(z) \equiv \hat{\psi}F(z). \tag{25}$$

Here $F(z)$ denotes a column vector composed of $F_n(z)$-elements. We substitute this expansion into equation (19) and multiply it on the left by matrix $\hat{\psi}^{-1}$. As a result we obtain

$$-\mathrm{i}\frac{\partial F}{\partial z} = \hat{\Lambda}F + \hat{P}F, \tag{26}$$

where it is designated

$$\hat{P} = \hat{\psi}^{-1}\hat{q}\hat{\psi}.$$

Thus, from the problem for vector $\varepsilon(z)$ we have passed to the problem for vector $F(z)$. This has the advantage that matrix $\hat{\Lambda}$ has the maximum possible simplest form.

Now we give the particular expressions for all the matrices and vectors introduced. To begin with, we describe some designations. We put

$$k_{z\mathrm{A}} = k_z^{(1,2)} \equiv k_x \tan\chi, \quad k_{z\mathrm{F}} = k_z^{(3)} \equiv \mathrm{i}k_t,$$
$$k_{z\mathrm{F}}^* = k_z^{(4)} \equiv -\mathrm{i}k_t.$$

Symbols A and F associate these quantities, respectively, with an Alfvén wave and a fast magnetosonic wave. If these values of k_z are substituted into the formulae $k_n = k_x \cos\chi + k_z \sin\chi$ and $k_\perp = \sqrt{k_n^2 + k_y^2}$, then we obtain

$$k_{n\mathrm{A}} = \frac{k_x}{\cos\chi}, \quad k_{\perp\mathrm{A}} = \sqrt{\frac{k_x^2}{\cos^2\chi} + k_y^2},$$

$$k_{n\mathrm{F}} = k_x \cos\chi + \mathrm{i}k_t \sin\chi, k_{\perp\mathrm{F}} = k_t \cos\chi + \mathrm{i}k_x \sin\chi,$$

and from two possible values of $k_\perp$ we have chosen those, for which $Re\, k_\perp > 0$.

Matrix $\hat{\psi}$ can be chosen in the form

$$\hat{\psi} = \begin{pmatrix} \frac{k_{n\mathrm{A}}}{k_{\perp\mathrm{A}}} & -\frac{(k_{n\mathrm{A}}^2 - k_y^2)\tan\chi}{k_{\perp\mathrm{A}}^2} & -\frac{k_y}{k_{\perp\mathrm{F}}} & -\frac{k_y}{k_{\perp\mathrm{F}}^*} \\ \frac{k_y}{k_{\perp\mathrm{A}}} & -\frac{2k_y k_{n\mathrm{A}}\tan\chi}{k_{\perp\mathrm{A}}^2} & \frac{k_{n\mathrm{F}}}{k_{\perp\mathrm{F}}} & \frac{k_{n\mathrm{F}}^*}{k_{\perp\mathrm{F}}^*} \\ 0 & \frac{k_{n\mathrm{A}}}{k_0} & -\mathrm{i}\frac{k_y}{k_0} & \mathrm{i}\frac{k_y}{k_0} \\ 0 & -\frac{k_y}{k_0} & -\mathrm{i}\frac{k_{n\mathrm{F}}}{k_0} & \mathrm{i}\frac{k_{n\mathrm{F}}^*}{k_0} \end{pmatrix}. \tag{27}$$

This yields the expression for matrix $\hat{\Lambda}$:

$$\hat{\Lambda} = \begin{pmatrix} k_{z\mathrm{A}} & k_{\perp\mathrm{A}}/\cos\chi & 0 & 0 \\ 0 & k_{z\mathrm{A}} & 0 & 0 \\ 0 & 0 & k_{z\mathrm{F}} & 0 \\ 0 & 0 & 0 & k_{z\mathrm{F}}^* \end{pmatrix}.$$

The constant S has received the value $S = k_{\perp\mathrm{A}}/\cos\chi$. Through a straightforward calculation one can make sure that all the relationships (23) are satisfied. In the subsequent treatment we shall need only the first column of matrix $\hat{P}$. It has the form

$$P^1 = \frac{k_0}{k_{\perp\mathrm{A}}\cos\chi}\begin{Bmatrix} \mathrm{i}\tan\chi\left(\frac{k_{n\mathrm{A}}}{k_{\perp\mathrm{A}}}\kappa_\perp - \frac{k_y}{k_{\perp\mathrm{A}}}\kappa_\mathrm{H}\right) \\ \mathrm{i}\kappa_\perp \\ \frac{1}{2}\left(\kappa_\mathrm{H} + \mathrm{i}\frac{k_y \sin\chi}{k_t}\kappa_\perp\right) \\ \frac{1}{2}\left(-\kappa_\mathrm{H} + \mathrm{i}\frac{k_y \sin\chi}{k_t}\kappa_\perp\right) \end{Bmatrix}. \tag{28}$$

In order to complete the formulation of the problem for vector $F(z)$, it is necessary to impose the boundary conditions for it. On the lower boundary of the ionosphere the role of them is played by the equalities (12) and (17). Instead of components $\bar{B}_t$, $\bar{E}_b$, and $\bar{B}_b$ in them, we shall use components $\bar{E}_n$, $\bar{E}_y$, $\bar{B}_y$, and $\bar{B}_n$ [for which it is necessary to employ the equalities (18)] and after that, with the help of relationship (25) we shall use the quantities $F_n(z)$. As a result, we obtain two boundary conditions

$$F_2(H) = 0, \tag{29}$$

$$(\lambda+1)F_3(H) + (\lambda-1)F_4(H) = 0. \tag{30}$$

One more boundary condition will be imposed on the upper boundary of the ionosphere. From equation (26) it follows that in the topside ionosphere, where matrix $\hat{P}$ virtually goes to zero,

$$F_3(z) = f_3 \exp[\mathrm{i}k_{z\mathrm{F}}(z-H)] = f_3\,\mathrm{e}^{-k_t(z-H)},$$
$$F_4(z) = f_4 \exp[\mathrm{i}k_{z\mathrm{F}}^*(z-H)] = f_4\,\mathrm{e}^{k_t(z-H)}.$$

Here f_3 and f_4 are some constants. We shall require the absence in the topside ionosphere of the upward growing solution $F_4(z)$. Physically, this means that the magnetosonic wave penetrating from the magnetosphere is absent, while the magnetosound field which is represented by the functions F_3 and F_4, is induced by an Alfvén wave incident from the magnetosphere during its interaction with the Hall and

Pedersen ionospheric layers. The field of such a magnetosound must decrease upwards. On the upper boundary we put

$$F_4(z_A) = 0. \tag{31}$$

Thus, for the system of four homogeneous equations (26) with four unknown functions $F_n(z)$ there exist three homogeneous boundary conditions (29)–(31). Consequently, the solution is determined up to a factor common for all functions F_n. Its value is then determined, when matching with the magnetospheric solution, by the Alfvén wave amplitude.

6. SOLUTION FOR THE ELECTROMAGNETIC FIELD IN THE IONOSPHERE

We shall solve the problem for vector $F(z)$, by using a modification of perturbation theory based on the smallness of matrix $\hat{P}$. The physical meaning of this smallness has been pointed out in the Introduction, namely that the frequency of the oscillations considered must be much less than ionospheric eigenfrequencies. Mathematically corresponding constraints will be formulated below.

When solving the system (26), it is convenient to introduce the new functions $f_n(z)$ defined by the equalities

$$F_n(z) = f_n(z)\exp[ik_z^{(n)}(z-H)]. \tag{32}$$

Let us try, at first, to apply a standard perturbation theory. In its zero order, by assuming $\hat{P} = 0$, we have the problem

$$f_1' = -i(k_{\perp A}/\cos\chi)f_2,\quad f_2' = f_3' = f_4' = 0,$$
$$f_2(H) = 0,\quad (\lambda+1)f_3(H)+(\lambda-1)f_4(H) = 0,$$
$$f_4(z_A) = 0.$$

It has the solution $f_1 = f = \text{const}$, $f_2 = f_3 = f_4 = 0$. By calculating, then, a first-order correction to this solution, it is easy to see that the correction to the function f_1 turns out to be not small compared with f_1 itself. The point here is that in the Pedersen layer of the lower ionosphere the function f_1, while changing little in magnitude, alters significantly its derivative, and in the topside ionosphere, while obeying the equation $f_1'' = 0$, it changes linearly with coordinate z and, owing to the high altitude of the ionosphere z_A alters its value significantly. Thus, a standard perturbation theory is inapplicable; but on the basis of the same formulae one can draw the conclusion that the inequalities

$$|f_2|, |f_3|,\quad |f_4| \ll |f_1|$$

hold throughout the entire ionosphere. It is this assumption that will form the basis for our varient of perturbation theory. It allows us to simplify considerably system (26), by retaining in the small term $\hat{P}F$ only the main component $P^1F_1 = P^1f_1\exp[ik_{zA}(z-H)]$. After that, the first pair of equations of the system splits out, and we begin our treatment by considering it.

We have the following equations

$$f_1' = i\frac{k_{\perp A}}{\cos\chi}f_2 - \frac{k_0\tan\chi}{k_{\perp A}\cos\chi}\left(\frac{k_{nA}}{k_{\perp A}}\kappa_\perp - \frac{k_y}{k_{\perp A}}\kappa_H\right)f_1, \tag{33a}$$

$$f_2' = -\frac{k_0}{k_{\perp A}\cos\chi}\kappa_\perp f_1, \tag{33b}$$

and the boundary condition

$$f_2(H) = 0. \tag{34}$$

We introduce the constant $f = f_1(H)$ which will play the role of a common factor for the entire solution. Besides, we denote

$$X_{P,H}(z) = \int_H^z \kappa_{P,H}(z')\,dz',$$

$$K_{P,H} = X_{P,H}(H+\Delta) \equiv \int_H^{H+\Delta}\kappa_{P,H}(z)\,dz.$$

Note that $K_{P,H} = 4\pi\Sigma_{P,H}/c$, where

$$\Sigma_{P,H} = \int_H^{H+\Delta}\sigma_{P,H}(z)\,dz$$

are the integral Pedersen and Hall ionospheric conductivities.

The right-hand side of equation (33b) contains the term $\kappa_\perp(z)f_1(z)$. In the lower ionosphere $\kappa_\perp \approx \kappa_P$, and the function $f_1(z)$, as will be shown in the following, changes little: $f_1(z) \approx f$. In the topside ionosphere the value of (21) for $\kappa_\perp$ is small and, by neglecting for the time being its difference from zero, one can see that in the entire ionosphere it is possible to put $\kappa_\perp(z)f_1(z) \approx \kappa_P(z)f$. From (33b) and (34) it then follows that

$$f_2(z) = -f\frac{k_0}{k_{\perp A}\cos\chi}X_P(z). \tag{35}$$

In the topside ionosphere, when $z > H+\Delta$, this equality yields

$$f_2(z) = -f\frac{k_0 K_P}{k_{\perp A}\cos\chi}. \tag{36}$$

Here the function $f_2(z)$ is a constant. From the obtained expressions it is apparent that for the inequality $|f_2| \ll |f_1|$ to be valid, it is necessary that the condition

$$k_0 K_P / k_{\perp A} \ll 1 \tag{37}$$

be satisfied. One can make sure that this same condition permits the second term on the right-hand side of equation (33a) to be omitted. After that, we get

$$f_1(z) = f[1 - i\frac{k_0}{\cos^2\chi}\int_H^z X_P(z')\,dz']. \tag{38}$$

It has been assumed above that in the lower ionosphere the function $f_1(z)$ changes little. From expression (38) it is evident that this is indeed the case if

$$k_0 K_P \Delta \ll 1. \tag{39}$$

The conditions (37) and (39) can be rewritten as inequalities

$$\omega \ll \frac{c^2 k_{\perp A}}{4\pi\Sigma_P}, \quad \frac{c^2}{4\pi\Sigma_P\Delta}, \tag{40}$$

which mean that the frequency of the oscillation considered is much lower than eigenmode frequencies of a low-frequency whistler in the lower ionosphere (Mazur, 1988). For the standing Alfvén waves of interest here these inequalities are satisfied. In the topside ionosphere, from (38), it follows to within an accuracy sufficient for us, that

$$f_1(z) = f\left[1 - i\frac{k_0 K_P(z-H)}{\cos^2\chi}\right]. \tag{41}$$

Note that for real values of the parameters, the value of $k_0 K_P z_A$ is of the order of, or higher than, unity, i.e. the variation of the function $f_1(z)$ in the topside ionosphere is not a small one. If we are interested in the function $f_1(z)$ itself rather than in the derivative, then the expression (41) can be used throughout the entire ionosphere, including the lower ionosphere because the difference between the expressions (38) and (41) is small.

In the preceding discussion we have ignored the fact that in the topside ionosphere the function $\kappa_\perp(z)$ does not go to zero but is equal to its collisionless value (21). If this is now taken into account, then using the iteration method, for $z > H+\Delta$ we obtain from (33b)

$$f_2(z) = -f\frac{k_0}{k_{\perp A}\cos\chi}\left\{K_P - \frac{i}{k_0}\int_{H+\Delta}^z k_A^2(z') \times\left[1 - i\frac{k_0 K_P(z'-H)}{\cos^2\chi}\right]dz'\right\}.$$

From this expression it is evident that terms which appear additionally, will be small compared with that already taken into account in (36) if

$$\left|\int_{H+\Delta}^{z_A} k_A^2(z)(z-H)\,dz\right| \ll 1.$$

This same inequality permits us to neglect analogous additional terms in the expression for $f_1(z)$ as well. By the order of magnitude, it is equivalent to

$$\left|\int_{H+\Delta}^{z_A} k_A(z)\,dz\right| \equiv \omega\left|\int_{H+\Delta}^{z_A}\frac{dz}{A(z)}\right| \ll 1. \tag{42}$$

From the physical point of view, condition (42) means that the frequency of the oscillations considered is much smaller than eigenfrequencies of the ionospheric Alfvén resonator (Polyakov and Rapoport, 1981). It is also satisfied for standing Alfvén waves.

Let us now consider the second pair of equations of the system (26) and the corresponding boundary conditions (30) and (31). We rewrite them for the functions f_n, by using explicit expressions for vector $\mathbf{P}^1$. We have the equations

$$f_3' = i\frac{k_0}{2k_{\perp A}\cos\chi}\left(\kappa_H + i\frac{k_y\sin\chi}{k_t}\kappa_P\right)f_1 \times\exp[(k_t + ik_x\tan\chi)(z-H)], \tag{43a}$$

$$f_4' = -i\frac{k_0}{2k_{\perp A}\cos\chi}\left(\kappa_H - i\frac{k_y\sin\chi}{k_t}\kappa_P\right)f_1 \times\exp[(-k_t + ik_x\tan\chi)(z-H)], \tag{43b}$$

and the boundary conditions

$$(\lambda+1)f_3(H) + (\lambda-1)f_4(H) = 0, \tag{44a}$$

$$f_4(z)|_{z\to\infty} = 0. \tag{44b}$$

In the last relationship we have transferred the upper boundary to infinity, bearing in mind that the ionospheric height z_A is much larger than a typical scale of variation of the function $f_4(z)$ which is determined by the parameters Δ and k_t^{-1}. As done above, it will be assumed here that $\kappa_H(z)f_1(z) \approx \kappa_H(z)f$ and $\kappa_P(z)f_1(z) \approx \kappa_P(z)f$. In other words, in equations (43) we put $f_1(z) = f$.

Integration of equation (43b), in view of the boundary condition (44b), yields

$$f_4(z) = if\frac{k_0}{2k_{\perp A}\cos\chi}\int_z^\infty\left[\kappa_H(z') - i\frac{k_y\sin\chi}{k_t}\kappa_P(z')\right] \times\exp[(-k_t + ik_x\tan\chi)(z'-H)]\,dz'.$$

This, in particular, gives

$$f_4(H) = \mathrm{i}f\frac{k_0}{2k_{\perp A}\cos\chi}\left[\bar{K}_H - \mathrm{i}\frac{k_y \sin\chi}{k_t}\bar{K}_P\right], \quad (45)$$

where it is designated

$$\bar{K}_{P,H} = \int_H^\infty \kappa_{P,H}(z)\exp[(-k_t + \mathrm{i}k_x\tan\chi)(z-H)]\,\mathrm{d}z. \quad (46)$$

From the equality (44a), by using an explicit expression for the quantity λ, we obtain

$$f_3(H) = -\frac{k_g-k_t}{k_g+k_t}\mathrm{e}^{-2k_tH}f_4(H) = -\mathrm{i}f\frac{k_0}{2k_{\perp A}\cos\chi}$$

$$\times\frac{k_g-k_t}{k_g+k_t}\mathrm{e}^{-2k_tH}\left(\bar{K}_H - \mathrm{i}\frac{k_y\sin\chi}{k_t}\bar{K}_P\right). \quad (47)$$

Finally, integration of equation (43a) yields

$$f_3(z) = f_3(H) + \mathrm{i}f\frac{k_0}{2k_{\perp A}\cos\chi}\int_H^z$$

$$\times\left[\kappa_H(z') + \mathrm{i}\frac{k_y\sin\chi}{k_t}\kappa_P(z')\right]$$

$$\times\exp[(k_t + \mathrm{i}k_x\tan\chi)(z'-H)]\,\mathrm{d}z'.$$

In summarizing the result of this section, we write out the expressions for all functions $F_n(z) = f_n(z)\exp[\mathrm{i}k_z^{(n)}(z-H)]$:

$$F_1(z) = f\left[1 - \mathrm{i}\frac{k_0K_P(z-H)}{\cos^2\chi}\right]$$

$$\times\exp[\mathrm{i}k_x(z-H)\tan\chi], \quad (48a)$$

$$F_2(z) = -f\frac{k_0X_P(z)}{k_{\perp A}\cos\chi}\exp[\mathrm{i}k_x(z-H)\tan\chi], \quad (48b)$$

$$F_3(z) = f_3(H)\,\mathrm{e}^{-k_t(z-H)} + \mathrm{i}\frac{k_0\,\mathrm{e}^{-k_t(z-H)}}{2k_{\perp A}\cos\chi}$$

$$\times\int_H^z\left[\kappa_H(z') + \mathrm{i}\frac{k_y\sin\chi}{k_t}\kappa_P(z')\right]$$

$$\times\exp[(k_t + \mathrm{i}k_x\tan\chi)(z'-H)]\,\mathrm{d}z', \quad (48c)$$

$$F_4(z) = \mathrm{i}f\frac{k_0}{2k_{\perp A}\cos\chi}\mathrm{e}^{k_t(z-H)}$$

$$\times\int_H^\infty\left[\kappa_H(z') - \mathrm{i}\frac{k_y\sin\chi}{k_t}\kappa_P(z')\right]$$

$$\times\exp[(-k_t + \mathrm{i}k_x\tan\chi)(z'-H)]\,\mathrm{d}z'. \quad (48d)$$

7. MATCHING OF SOLUTIONS IN THE IONOSPHERE AND MAGNETOSPHERE AND THE BOUNDARY CONDITION FOR ALFVEN WAVES

From the formulae of the preceding section it follows that in the topside ionosphere

$$F_1(z) = f\left[1 - \mathrm{i}\frac{k_0K_P(z-H)}{\cos^2\chi}\right]\mathrm{e}^{\mathrm{i}k_x(z-H)\tan\chi},$$

$$F_2(z) = -f\frac{k_0K_P}{k_{\perp A}\cos\chi}\mathrm{e}^{\mathrm{i}k_x(z-H)\tan\chi}, \quad (49)$$

$$F_3(z) \to 0, \quad F_4(z) \to 0. \quad (50)$$

These relationships are also equally applicable in the lower magnetosphere, merely because the boundary between these layers is a sufficiently arbitrary one. From relationship (50) it follows that a four-vector of electromagnetic field is given by the expression

$$\varepsilon(z) = \psi^1F_1(z) + \psi^2F_2(z). \quad (51)$$

Using explicit expressions for vectors ψ^1 and ψ^2 we then have

$$\bar{B}_y(z) = \frac{k_{nA}}{k_{\perp A}}B_A\,\mathrm{e}^{\mathrm{i}k_x(z-H)\tan\chi},$$

$$\bar{B}_n(z) = -\frac{k_y}{k_{\perp A}}B_A\,\mathrm{e}^{\mathrm{i}k_x(z-H)\tan\chi}. \quad (52)$$

Here we have designated

$$B_A = -fK_P/\cos\chi.$$

The quantity B_A is the oscillation amplitude of a disturbed magnetic field of an Alfvén wave in the topside ionosphere and in the lower magnetosphere. As one can see, in these layers it is a constant. The last equality can also be written as:

$$f = -B_A\cos\chi/K_P, \quad (53)$$

by expressing the constant f in terms of the Alfvén wave amplitude. The electric field of the Alfvén wave is given by the same expression (51). In view of relationship (53) and the inequalitity $|F_2| \ll |F_1|$, we have

$$\bar{E}_n(z) = \mathrm{i}B_A\frac{k_n}{k_{\perp A}}$$

$$\times\left[\frac{k_0(z-H)}{\cos\chi} + \mathrm{i}\frac{\cos\chi}{K_P}\right]\mathrm{e}^{\mathrm{i}k_x(z-H)\tan\chi}, \quad (54a)$$

$$\bar{E}_y(z) = \mathrm{i}B_A\frac{k_y}{k_{\perp A}}$$

$$\times\left[\frac{k_0(z-H)}{\cos\chi} + \mathrm{i}\frac{\cos\chi}{K_P}\right]\mathrm{e}^{\mathrm{i}k_x(z-H)\tan\chi}. \quad (54b)$$

In our previous papers (Leonovich and Mazur, 1989a,b, 1990) we have used the boundary condition for Alfvén waves on the ionosphere, without giving its derivation. We shall do this in the present paper because all formulae required for this have been obtained. The desired boundary condition is a relationship between the derivative of a disturbed field and itself on the upper boundary of the ionosphere. This condition has the most suitable form if the derivative is taken along a field line

$$\frac{\partial}{\partial l} = \cos\chi\frac{\partial}{\partial z} + \mathrm{i}k_x \sin\chi.$$

For calculating this derivative, the accuracy, to within which the expressions (52) are obtained, is insufficient. From them it follows that in the topside ionosphere and in the lower magnetosphere, and also when $z = z_{\mathrm{A}}$, we have

$$\frac{\partial \bar{B}_{n,y}}{\partial l} = 0. \qquad (55)$$

This equality can be considered to be a zero approximation for the desired boundary condition. It means that the magnetic field of the wave has an antinode on the ionosphere.

In order to obtain it to a desired accuracy, it is easiest to proceed from equations (33) which in the topside ionosphere and in the lower magnetosphere have the form

$$f'_1 = \mathrm{i}\frac{k_{\perp\mathrm{A}}}{\cos\chi}f_2, \quad f'_2 = \mathrm{i}\frac{k_{\mathrm{A}}^2}{k_{\perp\mathrm{A}}\cos\chi}f_1. \qquad (56)$$

From the equality (51) we have $\bar{B}_y = (k_{n\mathrm{A}}/k_0)F_2$, $\bar{B}_n = (-k_y/k_0)F_2$. By differentiating these equalities with respect to l and using the second equation of (56), we obtain

$$\frac{\partial \bar{B}_y}{\partial l} = \frac{k_{n\mathrm{A}}}{k_{\perp\mathrm{A}}}\frac{k_{\mathrm{A}}^2}{k_0}F_1, \quad \frac{\partial \bar{B}_n}{\partial l} = -\frac{k_y}{k_0}\frac{k_{\mathrm{A}}^2}{k_0}F_1. \qquad (57)$$

We substitute here the expression (48a). Using also relationships (52) and (53) we obtain

$$\frac{\partial \bar{B}_{n,y}}{\partial l} = \left(-\mathrm{i}\frac{k_{\mathrm{A}}^2\cos\chi}{k_0 K_{\mathrm{P}}} - k_{\mathrm{A}}^2\frac{z-H}{\cos\chi}\right)\bar{B}_{n,y}; \qquad (58)$$

and this is, in fact, the desired boundary condition for matching the solutions in the topside ionosphere and in the lower magnetosphere. Only the question as to where the boundary between these layers should be drawn, remains unclear, especially as the function k_{A}^2 (z) changes very rapidly in the topside ionosphere. In principle the answer is clear, namely that the boundary can be established at any height in the layers involved. The strong dependence of k_{A}^2 on z must not affect the results of application of the boundary condition.

In order to prove the last statement, we obtain the equation for $\bar{B}_{n,y}$. To accomplish this, we divide equations (57) by k_{A}^2, differentiate them with respect to l and use the first equation of (56). As a result, we get

$$\frac{\partial}{\partial l}\frac{1}{k_{\mathrm{A}}^2}\frac{\partial \bar{B}_{n,y}}{\partial l} + \bar{B}_{n,y} = 0. \qquad (59)$$

This equation coincides with equations for Alfvén waves in the magnetosphere given in our previous papers if the dispersion effects are neglected in them and if attention is confined to considering such a region in which the geomagnetic field can be considered homogeneous. Let us integrate equation (59) along a field line from point l_0 to point l, by assuming that both of them lie in the topside ionosphere or in the lower magnetosphere. This means that the distance between them is much less than the wavelength such that the functions $\bar{B}_{n,y}$ can be considered constants. We obtain

$$\frac{\partial \bar{B}_{n,y}(l)}{\partial l} = k_{\mathrm{A}}^2(l)\frac{1}{k_{\mathrm{A}}^2(l_0)}\frac{\partial \bar{B}_{n,y}(l_0)}{\partial l} - k_{\mathrm{A}}^2(l-l_0)\bar{B}_{n,y}(l_0).$$

This equality totally agrees with (58) if it is taken into consideration that during the motion along the field line $l-l_0 = (z-z_0)/\cos\chi$. It permits us to transfer the boundary condition imposed at point l_0, to any other point l on a given field line. It is clear that the solution must not change due to such a transfer.

Thus, the position of the boundary $z = z_{\mathrm{A}}$ is determined by considerations of convenience. It is appropriate to choose it as low as possible but such that above it the function $A(z)$ changes slowly. This approximately corresponds to the height of the maximum of the function $A(z)$ (see Fig. 1). Otherwise, the magnetospheric part of the field line would include portions with sharply different scales of variation of A, which would considerably complicate the solution of the equation for Alfvén waves in the magnetosphere.

The two terms in brackets on the right-hand side of the equality (58) play a substantially different role. Both of them are, in a sense, small. Since the functions $\bar{B}_{n,y}$ describe an Alfvén wave, a typical value of the derivative $\partial\bar{B}_{n,y}/\partial l \sim k_{\mathrm{A}}\bar{B}_{n,y}$. The two terms concerned are small compared with this value, which agrees quite well with the zeroth-order approximation (55). The second term is, in absolute value, of the order of or even greater than the first one, but since it is real, its role is unimportant. Taking it into account in the boundary condition leads to a small change of the standing wave frequency which is of no practical concern. The first term, however, plays a fundamentally important role by determining the standing wave damping decrement due to the dissipation in the ionosphere; therefore only it will be retained. As a result, the boundary condition can be written as

$$\left.\left(\frac{\partial \bar{B}_{n,y}}{\partial l}+\mathrm{i}\frac{\omega}{A}\frac{c^2\cos\chi}{4\pi\Sigma_{\mathrm{P}}A}\bar{B}_{n,y}\right)\right|_{z=z_{\mathrm{A}}}=0. \quad (60)$$

Note that in the equality it is easy to perform an inverse Fourier transform in k_x and k_y and to pass to the functions $\tilde{B}_{n,y}$, the boundary condition for which has exactly the same form.

It should be pointed out that the boundary condition of the form (60) for the case of a vertical geomagnetic field ($\chi = 0$) was obtained earlier for a simpler model of the medium in many papers (see reviews by Lyatsky and Maltsev, 1983; Southwood and Hughes, 1983). Our result means that the boundary condition virtually retains its form, despite the significant complication of the model (specifically, $\chi \neq 0$ and taking into account the presence of the topside ionosphere) and the treatment of waves with arbitrary k_x and k_y.

8. THE RELATIONSHIP BETWEEN ELECTROMAGNETIC FIELDS ON THE LOWER BOUNDARY OF THE MAGNETOSPHERE AND ON THE EARTH'S SURFACE

First of all, we shall obtain the relationships relating values of fields on the upper and lower boundaries of the ionosphere. Formulae for fields on the upper boundary have been obtained in the preceding section. From the equalities (34), (38), (45) and (47) as well as from the relationship (53) it follows that on the lower boundary

$$F_1(H) = -\frac{\cos\chi}{K_{\mathrm{P}}}B_{\mathrm{A}},$$

$$F_2(H) = 0,$$

$$F_3(H) = \mathrm{i}\frac{k_0}{2k_{\perp\mathrm{A}}}\frac{k_{\mathrm{g}}-k_{\mathrm{t}}}{k_{\mathrm{g}}+k_{\mathrm{t}}}\mathrm{e}^{-2k_{\mathrm{t}}H}\left(\frac{\bar{K}_{\mathrm{H}}}{K_{\mathrm{P}}}+\mathrm{i}\frac{k_y}{k_{\mathrm{t}}}\frac{\sin\chi\,\bar{K}_{\mathrm{P}}}{K_{\mathrm{P}}}\right)B_{\mathrm{A}},$$

$$F_4(H) = -\mathrm{i}\frac{k_0}{2k_{\perp\mathrm{A}}}\left(\frac{\bar{K}_{\mathrm{H}}}{K_{\mathrm{P}}}-\mathrm{i}\frac{k_y}{k_{\mathrm{t}}}\frac{\sin\chi\,\bar{K}_{\mathrm{P}}}{K_{\mathrm{P}}}\right)B_{\mathrm{A}}.$$

On substituting these values into formula (25), one can obtain expressions for the components $\bar{E}_n(H)$, $\bar{E}_y(H)$, $\bar{B}_y(H)$, and $\bar{B}_n(H)$ and using them, in view of the equalities (18), horizontal components of fields are obtainable. Simple, though cumbersome, calculations give

$$\bar{B}_{\mathrm{t}}(H) = \frac{k_{\mathrm{t}}}{k_{\perp\mathrm{A}}}\frac{\cosh k_{\mathrm{t}}H+(k_{\mathrm{t}}/k_{\mathrm{g}})\sinh k_{\mathrm{t}}H}{1+k_{\mathrm{t}}/k_{\mathrm{g}}}\mathrm{e}^{-k_{\mathrm{t}}H}\times\left(\frac{\bar{K}_{\mathrm{H}}}{K_{\mathrm{P}}}-\mathrm{i}\frac{k_y}{k_{\mathrm{t}}}\frac{\sin\chi\,\bar{K}_{\mathrm{P}}}{K_{\mathrm{P}}}\right)B_{\mathrm{A}},$$

$$\bar{E}_{\mathrm{b}}(H) = -\mathrm{i}\frac{k_0}{k_{\perp\mathrm{A}}}\frac{\sinh k_{\mathrm{t}}H+(k_{\mathrm{t}}/k_{\mathrm{g}})\cosh k_{\mathrm{t}}H}{1+k_{\mathrm{t}}/k_{\mathrm{g}}}\mathrm{e}^{-k_{\mathrm{t}}H}\times\left(\frac{\bar{K}_{\mathrm{H}}}{K_{\mathrm{P}}}-\mathrm{i}\frac{k_y}{k_{\mathrm{t}}}\frac{\sin\chi\,\bar{K}_{\mathrm{P}}}{K_{\mathrm{P}}}\right)B_{\mathrm{A}},$$

$$\bar{E}_{\mathrm{t}}(H) = -\mathrm{i}\frac{k_{\mathrm{t}}\cos\chi}{k_{\perp\mathrm{A}}}\frac{1}{K_{\mathrm{P}}}B_{\mathrm{A}},$$

$$\bar{B}_{\mathrm{b}}(H) = 0. \quad (61)$$

It is easy to see that these expressions satisfy the boundary conditions (12) and (17). Finally, from formulae (61) and from the results of Section 4 it follows that on the Earth's surface

$$\bar{B}_{\mathrm{t}}(0) = \frac{k_{\mathrm{t}}}{k_{\perp\mathrm{A}}}\frac{1}{1+k_{\mathrm{t}}/k_{\mathrm{g}}}\mathrm{e}^{-k_{\mathrm{t}}H}\times\left(\frac{\bar{K}_{\mathrm{H}}}{K_{\mathrm{P}}}-\mathrm{i}\frac{k_y}{k_{\mathrm{t}}}\frac{\sin\chi\,\bar{K}_{\mathrm{P}}}{K_{\mathrm{P}}}\right)B_{\mathrm{A}}, \quad (62\mathrm{a})$$

$$\bar{E}_{\mathrm{b}}(0) = -\mathrm{i}\frac{k_0}{k_{\mathrm{g}}}\bar{B}_{\mathrm{t}}(0) = -\mathrm{i}\frac{k_{\mathrm{t}}}{k_{\perp\mathrm{A}}}\frac{k_0}{k_{\mathrm{t}}+k_{\mathrm{g}}}\mathrm{e}^{-k_{\mathrm{t}}H}\times\left(\frac{\bar{K}_{\mathrm{H}}}{K_{\mathrm{P}}}-\mathrm{i}\frac{k_y}{k_{\mathrm{t}}}\frac{\sin\chi\,\bar{K}_{\mathrm{P}}}{K_{\mathrm{P}}}\right)B_{\mathrm{A}}, \quad (62\mathrm{b})$$

$$\bar{E}_{\mathrm{t}}(0) = 0, \quad \bar{B}_{\mathrm{b}}(0) = 0. \quad (62\mathrm{c})$$

The results obtained, in total, solve the problem of the electromagnetic field induced in ground layers by an Alfvén wave incident from the magnetosphere, provided that the wave is a separate Fourier-harmonic in time and in horizontal coordinates $\sim\exp(\mathrm{i}k_x x+\mathrm{i}k_y y-\mathrm{i}\omega t)$. Indeed, formulae (48) and (25), together with the equality (53), define the electromagnetic field in the ionosphere and in the lower magnetosphere, formulae (10), (11), (13) and (16), together with (61) or (62), define the field in the atmosphere, and formulae (8) and (62) define the field in the Earth. By performing an inverse Fourier-transform in k_x, k_y and ω, one can obtain the electromagnetic field distribution in ground layers for an Alfvén wave with an arbitrary space-time structure. We have accomplished this procedure for the field on the Earth's surface, i.e. the most important case for practical purposes.

Let us limit our attention to the case of a highly conducting Earth, $k_{\mathrm{t}}/k_{\mathrm{g}} \to 0$. From the relationships

(62) and (52) it follows that

$$\bar{B}_x(k_x, k_y, 0, \omega) = \cos\chi \cdot \bar{R}(k_x, k_y)\cdot \bar{B}_y(k_x, k_y, z_A, \omega), \quad (63a)$$

$$\bar{B}_y(k_x, k_y, 0, \omega) = -\bar{R}(k_x, k_y)\cdot \bar{B}_n(k_x, k_y, z_A, \omega), \quad (63b)$$

$$\bar{B}_z(k_x, k_y, 0, \omega) = \mathrm{i}\frac{\bar{R}(k_x, k_y)}{k_g}[k_y \bar{B}_n(k_x, k_y, z_A, \omega) - k_x \cos\chi \bar{B}_y(k_x, k_y, z_A, \omega)]. \quad (63c)$$

Here

$$\bar{R}(k_x, k_y) = \left(\frac{\bar{K}_H}{\bar{K}_P} - \mathrm{i}\frac{k_y \sin\chi}{k_t}\frac{\bar{K}_P}{\bar{K}_P}\right) \times \exp[-k_t H - \mathrm{i}k_x(z_A - H)\tan\chi]$$
$$\equiv \frac{1}{\Sigma_P}\int_0^\infty \left[\sigma_H(z) - \mathrm{i}\frac{k_y \sin\chi}{k_t}\sigma_P(z)\right] \times \exp[-k_t z + \mathrm{i}k_x(z - z_A)\tan\chi]\,\mathrm{d}z. \quad (64)$$

Bearing in mind that the functions $\sigma_H(z)$ and $\sigma_P(z)$ are substantially different from zero only in the lower ionosphere, integration over z formally extends to the entire interval $(0, \infty)$. As a result, it becomes apparent that these formulae lack the dependence on height H, i.e. an artificial boundary between the atmosphere and ionosphere.

Results of an earlier work devoted to an analytic investigation of the passage of the Alfvén wave field to the earth through a horizontally-homogeneous ionosphere, are particular cases of formulae (63) and (64). In one of them it is assumed that the geomagnetic field is a vertical one, $\chi = 0$, and the horizontal wavelength of the disturbance is much larger than the typical thickness of the ionospheric conducting layer, $k_t\Delta \ll 1$ (Hughes and Southwood, 1976a). Then

$$\bar{R}(k_x, k_y) = \frac{\Sigma_H}{\Sigma_P}\mathrm{e}^{-k_t H}. \quad (65a)$$

In another particular case the geomagnetic field is assumed inclined, $\chi \neq 0$ but independent of the coordinate y (i.e. it is assumed that $k_y = 0$) and also the condition $k_t\Delta \ll 1$ is supposed (Hughes, 1974). In this case

$$\bar{R}(k_x, 0) = \frac{\Sigma_H}{\Sigma_P}\exp[-|k_x|H - \mathrm{i}k_x x_A], \quad (65b)$$

where it is designated $x_A = (z_A - H)\tan\chi$. Incidentally, formula (65a) can be considered the particular case of formula (65b) because when $\chi = 0$ the horizontal wave vector $\mathbf{k}_t$ can, without loss of generality, be directed along axis x.

The conditions adopted when deriving the relationships (65) place considerable constraints on the possibilities of applying them to the real oscillations. The requirement $k_y = 0$ forces one to consider only toroidal (or close to them) oscillations of the magnetosphere. The condition $k_t\Delta \ll 1$ is extremely burdensome. In connection with this, we want to note that the presence of the factor $\exp(-k_t H)$ in formulae (65), which represents the field attenuation on the Earth for short-wavelength oscillations with $k_t \gtrsim H^{-1}$, is, in fact, an excess of accuracy. The point here is that the parameters Δ and H are the values of the same order of magnitude; therefore, the condition $k_t\Delta \ll 1$ simultaneously means also $k_t H \ll 1$. Thus, formulae (65) should be written in a still simpler form

$$\bar{R} = \frac{\Sigma_H}{\Sigma_P}\exp(-\mathrm{i}k_x x_A),$$

but this expression is inapplicable for transversally-small-scale oscillations.

The general expression (64) we have obtained is free from these limitations. When $k_t\Delta \gtrsim 1$, the value of the function $\bar{R}(k_x, k_y)$ is not determined only by integral conductivities but depends substantially on the profile of the functions $\sigma_H(z)$ and $\sigma_P(z)$. Besides, for waves with $k_y \neq 0$ (and when $\chi \neq 0$), in the expression for the function $\bar{R}(k_x, k_y)$ there appears a fundamentally new term which shows that the penetration of an Alfvén wave to the Earth can be due not only to Hall conductivity but also to Pedersen conductivity.

Let us perform in formulae (63) an inverse Fourier-transform in k_x, k_y and ω. We denote

$$R(\xi, \eta) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}k_x\int_{-\infty}^{\infty}\mathrm{d}k_y \times \bar{R}(k_x, k_y)\,\mathrm{e}^{\mathrm{i}k_x\xi + \mathrm{i}k_y\eta}. \quad (66)$$

Then

$$B_x(x, y, 0, t) = \cos\chi\int_{-\infty}^{\infty}\mathrm{d}x'\int_{-\infty}^{\infty}\mathrm{d}y' \times R(x - x', y - y')B_y(x', y', z_A, t), \quad (67a)$$

$$B_y(x, y, 0, t) = -\int_{-\infty}^{\infty}\mathrm{d}x'\int_{-\infty}^{\infty}\mathrm{d}y' \times R(x - x', y - y')B_n(x', y', z_A, t), \quad (67b)$$

We shall not give here the more unwieldy expression for $B_z(x, y, 0, t)$. By taking integrals over k_x and k_y in (66), we obtain a relatively simple expression for the nucleus of integral transformations:

$$R(\xi,\eta) = \frac{1}{2\pi\Sigma_P} \times \int_0^\infty \frac{z\sigma_H(z) + \eta\sigma_P(z)\sin\chi}{[z^2 + (\xi + (z - z_A)\tan\chi)^2 + \eta^2]^{3/2}}\,dz. \quad (68)$$

It is easy to see that exactly the same formulae also occur for Fourier-transforms in time $\tilde{B}_{x,y}(x, y, z, \omega)$. Basically, formulae (67) and (68) solve the problem of the space-time structure of the oscillation field on the terrestrial surface if it is known on the lower boundary of the magnetosphere. They also yield simple particular cases as obtained in the above-cited references. These particular cases are to the same extent limited compared with the general relationship (68) as are the expressions (65) compared with (64).

The nucleus of integral transformation $R(x-x', y-y')$ in formulae (67) can be treated as a disturbance on the ground produced by a source concentrated in the magnetosphere on one field line passing through point x', y' at the ionosphere–magnetosphere interface, i.e. at height $z = z_A$. A qualitative idea of this function may be obtained, by assuming that the thickness of the layer in which the Hall and Pedersen conductivities of the ionosphere are concentrated, is much smaller than the height of an isotropic atmosphere: $\Delta \ll H$. From (68) it follows that

$$R(\xi,\eta) = \frac{1}{2\pi}\left(H\frac{\Sigma_H}{\Sigma_P} + \eta\sin\chi\right) \times [H^2 + (\xi - (z_A - H)\tan\chi)^2 + \eta^2]^{-3/2}. \quad (69)$$

This expression can be applied in formulae (67) if a typical scale of the functions $B_{n,y}$ in coordinate x is much larger than $\Delta\tan\chi$ (which is far from being always satisfied). A typical scale of the function $R(\xi,\eta)$ in both variables is of the order of the atmospheric height, and this is particularly obvious for expression (69). This means that small-scale details in the oscillation field distribution at the lower edge of the magnetosphere are smoothed out when transported to the Earth, i.e. a typical scale of such details on the earth's surface cannot be smaller than H.

We wish to note in conclusion one particular case of formulae (67) and (68) which is important for applications. If a typical scale of the magnetospheric field $B_{n,y}(x, y, z_A, t)$ in the variable y is much larger than H, then

$$B_x(x, y, 0, t) = \cos\chi \int_{-\infty}^{\infty} dx' P(x - x') B_y(x', y, z_A, t), \quad (70a)$$

$$B_y(x, y, 0, t) = -\int_{-\infty}^{\infty} dx' P(x - x') B_n(x', y, z_A, t), \quad (70b)$$

where

$$P(\xi) = \int_{-\infty}^{\infty} R(\xi,\eta)\,d\eta = \frac{1}{\pi\Sigma_P}\int_0^\infty \frac{z\sigma_H(z)\,dz}{z^2 + [\xi + (z - z_A)\tan\chi]^2}. \quad (71)$$

9. CONCLUSIONS

Let us summarize the main results of this study.

(1) Within the framework of a horizontally-homogeneous model of ground layers adopted here, which realistically represents its vertical stratification, we have determined the spatial structure of the electromagnetic field induced in the ionosphere, the atmosphere and on the ground by low-frequency Alfvén oscillations of the magnetosphere. The problem has been fully solved for separate Fourier-harmonics, that is oscillations which are monochromatic waves with a given frequency ω and have definite values of the component of the horizontal vector (k_x, k_y) in the coordinates x and y.

(2) Based on this result, using the inverse Fourier-transform in ω, k_x and k_y it is possible to determine the space-time structure of the field in the ground layers for an arbitrary Alfvén wave in the magnetosphere. Such a problem has been solved virtually for the most important case of the field on the terrestrial surface. Formulae have been obtained which represent a disturbed magnetic field on the ground in terms of the Alfvén wave field on the lower edge of the magnetosphere. These formulae generalize substantially the results of previous work, enabling the transversally-small-scale oscillations to be considered with the most general dependence on the coordinates x and y. From them it follows that, for such oscillations, the ionosphere cannot be regarded as a thin film whose properties are determined by its integral conductivity, but an important role is played by the height distribution of conductivities. From simple physical considerations it is clear that the horizontal

inhomogeneity of the medium is needed on a scale comparable with the horizontal scale of the oscillation field (i.e. of order H). A horizontal inhomogeneity with a scale much larger than H can be readily introduced into theory; in order to do this, it is necessary to use local values of the parameters in formulae.

(3) Within the framework of the same model of the medium, we have obtained a boundary condition for Alfvén waves on the ionosphere–magnetosphere boundary. It includes all information on the ionosphere and on lower-lying layers needed for solving the problem of low-frequency Alfvén oscillations of the magnetosphere, and permits this problem to be solved, without recourse to considering oscillations in the ground layers. The obtained condition is also a generalization of previous results for the case of transversally-small-scale oscillations with an arbitrary dependence on transverse coordinates.

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APPENDIX

The equation governing the magnetosonic wave field in the axisymmetrical magnetosphere was obtained in our previous paper (Leonovich and Mazur, 1989a). It is not possible to find its analytical solution, unlike the equation for an Alfvén wave. In order to get a qualitative idea of the solution, we now examine a simplified equation and a simplified model of the medium. For small-scale magnetosonic waves, which have a spatial scale much less than the magnetospheric inhomogeneity scale, the equation takes the form

$$\Delta \hat{B}_3 + \frac{\omega^2}{A^2}\hat{B}_3 = 0. \tag{A1}$$

On the limit of applicability it also qualitatively correctly describes the large-scale magnetosound. It will be assumed that the Alfvén velocity depends only on the radial coordinate r measured from the terrestrial center: $A = A(r)$. Such a model makes it possible to describe the main property of the Alfvén velocity, namely its manyfold increase from the magnetospheric periphery to the Earth.

The solution of equation (A1) can be sought in the form

$$\hat{B}_3 = R(r)\, Y_{l,m}(\theta,\varphi)\, e^{im\varphi},$$

where φ is the azimuthal angle, θ the polar angle measured from the equator, and $Y_{l,m}(\theta,\varphi)$ is a spherical function that obeys the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial Y}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + l(l+1)Y = 0,$$

and the radial function $R(r)$ is defined by the equation

$$\frac{d^2 rR}{dr^2} + \left[\frac{\omega^2}{A^2} - \frac{l(l+1)}{r^2}\right] rR = 0. \tag{A2}$$

Actually observed magnetosonic oscillations, which cause a resonance excitation of standing Alfvén waves in the magnetosphere, are dominated by modes with azimuthal wavenumber $m = 3$–7 (Takahashi and McPherron, 1984). At a given m the wavenumber l satisfies the inequality $l \geqslant |m|$, and the solution with $l = |m|$ decreases the most slowly to the centre. The corresponding spherical function at minimum values of $m = \pm 3$ and $l = 3$ has the form

$$Y_{3,\pm 3} = \pm i\sqrt{\frac{35}{64\pi}}\cos^3(\theta)\, e^{\pm i3\varphi}.$$

The position of the spherical surface $r = \bar{r}$, which is the separation boundary between the transparency ($r > \bar{r}$) and opacity ($r < \bar{r}$) regions, is defined by the equation

$$\frac{\omega^2}{A^2(\bar{r})} - \frac{l(l+1)}{\bar{r}^2} = 0.$$

Simple estimations indicate that this surface lies farther from the Earth as compared with the resonance surface $r = r_0$

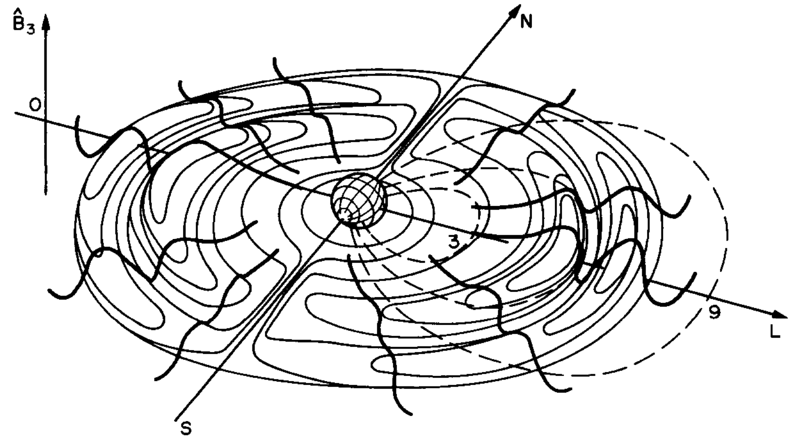

Fig. A1. A conceptual representation of the magnetosonic field in the meridional cross-section of the magnetosphere.
In the outer magnetosphere (transparency region) the field has an oscillatory character and in the inner region (opacity region) it decreases rapidly in the earthward direction. The dashed lines represent the geomagnetic field lines lying in the cross-section plane.

defined by the condition $\omega = \Omega_N(r_0)$. In other words, the resonance surface is located in the opacity region for magnetosound. Taking for the resonance surface $r_0 = (3\text{–}7)R_E$ we obtain for $l = 3$ an estimate of $\bar{r} = (5\text{–}10)R_E$. The radial function in the opacity region decreases inward the magnetosphere as $R \sim r^{-l}$. For the field on the lower magnetospheric edge, we have an estimate

$$R(R_E) \sim \left(\frac{\bar{r}}{R_E}\right)^{-l} R(\bar{r}),$$

that is, the magnetosound amplitude decreases at least by $5^3 \sim 100$ times. In other words, the magnetosonic wave virtually does not reach the ionosphere. Qualitatively, the magnetosound wave field is portrayed in Fig. A1.