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AN ELECTROMAGNETIC FIELD, INDUCED ON THE EARTH'S SURFACE BY STANDING ALFVEN WAVES IN THE MAGNETOSPHERE: THE PARTICULAR KINDS OF OSCILLATIONS

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Abstract—Based on a general theory of penetration of low-frequency Alfvén oscillations from the magnetosphere to the Earth as developed in our previous paper, we have solved the problem of the distribution, over the Earth's surface, of a disturbed magnetic field for particular kinds of standing Alfvén waves. Four kinds of oscillations have been considered, namely toroidal Alfvén waves excited by a monochromatic magnetosonic wave, by a sudden magnetosonic impulse and by a stochastic magnetosound as well as poloidal monochromatic Alfvén waves.

1. INTRODUCTION

The present paper is a direct continuation of our previous article (Leonovich and Mazur, 1991). Based on a general theory developed there we solve the problem of the distribution, over the Earth's surface, of an electromagnetic field induced by the particular kinds of Alfvén oscillations of the magnetosphere, i.e. standing Alfvén waves. A problem of such a kind was solved earlier by Hughes and Southwood (1976a,b). They used an extremely simple ("naïve" as they call it) magnetospheric model in the form of a flat layer of perpendicularly inhomogeneous plasma bounded at its ends by parallel planes modelling the conjugate hemispheres of the Earth.

The model we have used is more close to reality and includes a model of an axisymmetric magnetosphere and of ground layers which are described in detail by Leonovich and Mazur (1989a, 1991). We are using the solutions for standing Alfvén waves in the magnetosphere, which were obtained earlier (Leonovich and Mazur, 1989a,b, 1990) in terms of these models. In the present paper these solutions will be extended from the magnetosphere to the Earth's surface. We shall consider the following kinds of standing Alfvén waves: toroidal oscillations excited by a monochromatic magnetosound, by a sudden magnetosonic impulse and by a stochastic magnetosound as well as monochromatic poloidal Alfvén waves. These solutions qualitatively represent the main properties of the most important types of real oscillations of the Earth's magnetosphere. Thus, the present study completes in a sense, the above-cited series of our investigations of standing Alfvén waves in an axisymmetric magnetosphere; because it is closely associated with all papers of this series, we shall extensively use here the designations introduced in them.

2. A STANDING ALFVEN WAVE IN THE MAGNETOSPHERE

In order to describe an axisymmetric magnetosphere, we have used in our earlier papers a curvilinear orthogonal coordinate system x^1, x^2, x^3, in which the surfaces x^1 = const coincide with magnetic shells, and coordinates x^2 and x^3 specify a field line on a given shell and a point on a given field line, respectively (see Fig. 1). Symmetry about the invert of the axial axis (i.e. North–South symmetry) is not assumed here. Therefore, the equatorial surface, which is a separatrix for the coordinate surfaces x^3 = const, is generally not a flat one. We put $x^3 = 0$ on it. x^3_+ and x^3_- denote coordinates of intersection of the field line with upper boundaries of the ionospheres of the conjugate hemispheres. These quantities are functions of the magnetic shell: $x^3_\pm = x^3_\pm(x^1)$. The diagonal components of the metric tensor are denoted by g_i ($i = 1,2,3$) and its determinant is $g = g_1 g_2 g_3$.

In the ground region of space, whose size is small compared with the Earth's size, the coordinate lines of this curvilinear coordinate system approximately coincide with those of the Cartesian system (n, y, l) described in our previous paper. The quantities g_i can be considered constant here, and we denote them by $g_i^{(\pm)}$ (from here on we shall consider simultaneously the two conjugate hemispheres). Let us place the ori-

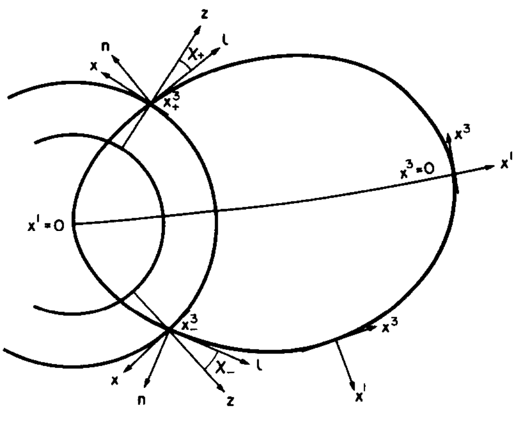

FIG. 1. THE COORDINATE SYSTEMS IN THE MERIDIONAL PLANE $x^2 = \text{CONST}$ AS USED IN THIS PAPER: (x^1, x^3)—CURVILINEAR ORTHOGONAL COORDINATE SYSTEM TIED WITH GEOMAGNETIC FIELD LINES IN THE MAGNETOSPHERE; (n, l)—LOCALLY-EUCLIDEAN COORDINATE SYSTEM, WHOSE AXIS l IS TANGENT TO THE FIELD LINE NEAR THE IONOSPHERE; AND (x, z)—EUCLIDEAN COORDINATE SYSTEM, WHOSE AXIS z IS VERTICAL.
Points of intersection of the field line with the upper boundary of the ionosphere are denoted by x^3_+ and x^3_-.

gin of the coordinates of the system (n, y, l) at point $(\bar{x}^1, \bar{x}^2, x^3_\pm)$, where $(\bar{x}^1, \bar{x}_2)$ is a certain given field line, and $\bar{x}^3_\pm = x^3_\pm(\bar{x}^1)$. For the region under consideration, we then have

$$n = \sqrt{g_1^{(\pm)}}(x^1 - \bar{x}^1), \quad y = \sqrt{g_2^{(\pm)}}(x^2 - \bar{x}^2),$$
$$l = \mp\sqrt{g_3^{(\pm)}}(x^3 - \bar{x}^3_\pm),$$

and

$$x^1 - \bar{x}^1 = \frac{\cos\chi_\pm}{\sqrt{g_1^{(\pm)}}}x - \frac{\sin\chi_\pm}{\sqrt{g_1^{(\pm)}}}(z - z_A);$$
$$x^3 - \bar{x}^3_\pm = \mp\frac{\sin\chi_\pm}{\sqrt{g_3^{(\pm)}}}x \mp \frac{\cos\chi_\pm}{\sqrt{g_3^{(\pm)}}}(z - z_A).$$

From the last formulae it follows, in particular, that on the ionosphere–magnetosphere boundary $(z = z_A)$:

$$x^1 = \bar{x}^1 + \frac{\cos\chi_\pm}{\sqrt{g_1^{(\pm)}}}x, \quad x^3 = \bar{x}^3_\pm \mp \frac{\sin\chi_\pm}{\sqrt{g_3^{(\pm)}}}x \equiv x^3_\pm(x^1).$$

Let $B_i(x^1, x^2, x^3, t)$ be covariant components of a disturbed magnetic field and let $\tilde{B}_i(x^1, x^2, x^3, \omega)$ be their Fourier-harmonics in time. Physical components of all vectors will be denoted by a "cap" over the letter, specifically $\hat{B}_i = B_i/\sqrt{g_i}$. On the ionosphere–magnetosphere boundary we have

$$B_{n,y}(x, y, z_A, t)$$
$$= \hat{B}_{1,2}\left(\bar{x}^1 + \frac{\cos\chi_*}{\sqrt{g_1^*}}x, \bar{x}^2 + \frac{1}{\sqrt{g_2^*}}y, x^3{}_*, t\right). \quad (1)$$

An asterisk as the index will henceforth mean that a value of the quantity on the ionosphere–magnetosphere boundary for any of the two conjugate hemispheres is taken. A formula analogous to (1) is valid also for the Fourier-transforms $\tilde{B}_{n,y}(x, y, z_A, \omega)$. These latter ones can be expanded in terms of spatial Fourier-harmonics:

$$\tilde{B}_{n,y}(x, y, z_A, \omega) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y$$
$$\times \bar{B}(k_x, k_y, z_A, \omega)\exp(ik_x x + ik_y y).$$

It is the functions $B_{n,y}(x, y, z_A, t)$ and $\bar{B}_{n,y}(k_x, k_y, z_A, \omega)$ introduced in this way which are involved in formulae (63), (67) and (70) of our previous paper.

3. THE GROUND FIELD OF A STANDING ALFVEN WAVE EXCITED BY A MONOCHROMATIC MAGNETOSOUND

Leonovich and Mazur (1989a,b) constructed an analytic theory of resonance excitation of standing Alfvén waves by a magnetosound in an axisymmetric magnetosphere which gives relatively simple formulae describing the space-time structure of the disturbance field in the magnetosphere. According to this theory, the excited Alfvén waves are nearly toroidal ones, i.e. the azimuthal component of the disturbanced magnetic field is much larger than a normal one, $\hat{B}_2 \gg \hat{B}_1$; accordingly, the spatial scale of the oscillation in the direction normal to the magnetic shell is much less than the azimuthal one. For that reason, we shall confine ourselves to examining the azimuthal component $\hat{B}_2$ in the magnetosphere and, correspondingly, the meridional component B_x on the ground. Having a property of toroidality, Alfvén waves, in other respects, depend strongly on the character of magnetosonic oscillations. In this and in the next two sections we shall consider three different examples.

Monochromatic magnetic sound with a given frequency ω excites an Alfvén wave in a narrow neighbourhood of the resonance magnetic shell, whose position is defined by the equation $\Omega_N(x^1) = \omega$, where Ω_N is the frequency of the N-th harmonic of toroidal eigen-oscillations of the magnetosphere. We shall restrict ourselves to the most typical case when the resonance shell lies in the region of monotonic variation of the function $\Omega_N(x^1)$, where near the resonance surface one can use the representation

$$\Omega_N(x^1) = \bar{\Omega}_N\left(1 - \frac{x^1 - \bar{x}^1}{l_N}\right). \quad (2)$$

Here $\bar{x}^1$ is a coordinate of the resonance surface, on which $\bar{\Omega}_N = \omega$.

According to results reported by Leonovich and Mazur (1989a), the Alfvén wave field is represented by the expression

$$\hat{B}_2(x^1, x^2, x^3, t) = M(x^1, x^2) \times \hat{H}_N(x^3)\phi\left(\frac{x^1 - \bar{x}^1}{b} + \mathrm{i}\varepsilon\right)\mathrm{e}^{-\mathrm{i}\omega t}. \quad (3)$$

The function

$$\hat{H}_N(x^3) = H_N(\bar{x}^1, x^3)/\sqrt{g_2(\bar{x}^1, x^3)}$$

represents the longitudinal structure of a standing wave. Index N denotes the harmonic number and is equal to the number of nodes of the function on a field line. When $N \gg 1$, one can apply formulae of the WKB approximation:

$$\hat{H}_N(x^3) = \left(\frac{2}{At_A}\right)^{1/2} \frac{1}{(g_1 g_2)^{1/4}} \times \cos\left(\Omega_N \int_{x^3_-}^{x^3} \frac{\sqrt{g_3}\, \mathrm{d}x^{3'}}{A}\right). \quad (4)$$

This formula represents qualitatively precisely $\hat{H}_N(x^3)$ even if $N \sim 1$. The function $\phi(\xi)$ represents the field structure along the normal to the magnetic shell. It has the following integral representation

$$\phi(\xi) = -\mathrm{i}\int_0^\infty \mathrm{d}v \exp\left(-\mathrm{i}\frac{v^3}{3} + \mathrm{i}v\xi\right), \quad (5)$$

which gives the asymptotic representations

$$\phi(\xi) = \begin{cases} -\dfrac{\pi^{1/2}}{\xi^{1/4}} \exp\left(\dfrac{2}{3}\mathrm{i}\xi^{3/2} + \mathrm{i}\dfrac{\pi}{4}\right) & \text{when } \xi \to \infty, \quad (6a) \\[2ex] \dfrac{1}{\xi} & \text{when } \xi \to -\infty. \quad (6b) \end{cases}$$

This last formula is applicable not only when $\xi \to -\infty$ but also throughout the entire sector $0 < \arg \xi < 4\pi/3$ of a complex plane ξ, in particular for imaginary positive values of ξ. A more detailed description of the function $\phi(\xi)$ is given in the Appendix of a paper by Leonovich and Mazur (1989a). The parameter b defines the spatial scale of oscillations across the magnetic shells. It is given by the formula $b = 2^{-1/3}\rho_N^{2/3} l_N^{1/3}$, where ρ_N is the dispersion length. The dimensionless parameter $\varepsilon = (l_N/b)(\gamma_N/\Omega_N)$ characterizes the relative role of the dispersion and dissipation in the oscillation field structure. When $\varepsilon \ll 1$, the dispersion is dominant. According to formula (6a), the oscillation field in the transverse direction is then an escaping wave. When $\varepsilon \gg 1$, the transverse structure is determined by the dissipation. In this case one can apply formula (6b) which gives a well-known expression $\phi \sim (x^1 - \bar{x}^1 + \mathrm{i}\varepsilon b)^{-1}$. When $\varepsilon \sim 1$, the roles of the dispersion and dissipation are comparable. Finally, the function $M(x^1, x^2)$ characterizes the Alfvén wave amplitude and is specified by the magnetosound field. A typical scale of its variation in the variables x^1 and x^2 coincides with the one for the magnetic sound and is comparable with the size of the magnetosphere, i.e. is much larger than the value of b.

All of the above parameters are functions of the magnetic shell. If we restrict ourselves to the dayside magnetosphere, then with McIlwain's parameter varying from $L = 1.3$–1.5 to $L = 7$–10, they vary in the range

$$l_N = (10^3\text{–}3 \cdot 10^4)\ \text{km},$$

$$\rho_N = (0.1\text{–}30)\ \text{km}, \quad \text{and} \quad \gamma_N/\Omega_N = 10^{-1}\text{–}10^{-3}.$$

From this we have $b = 3$–300 km and $\varepsilon = 10^2$–10^{-1}.

From formulae (1) and (3), on the ionosphere–magnetosphere boundary, we get

$$B_y(x, y, z_A, t) = B_*(x, y)\phi\left(\frac{x}{b_x} + \mathrm{i}\varepsilon\right)\mathrm{e}^{-\mathrm{i}\omega t}, \quad (7)$$

which is designated

$$B_* = M(x^1, x^2)\hat{H}_N(x^3_*)\cos\chi, \quad b_x = b\sqrt{g_1^*}/\cos\chi. \quad (8)$$

The quantity b_x is a typical transverse scale b projected onto the ionosphere. A typical range of its variation is $b_x = 1$–30 km. The field on the ground can be calculated by using formulae (70) of our previous paper; however, it is easier to do this by passing to spatial Fourier-harmonics. From (5) and (7), by neglecting the weak dependence of b_* on x and y, it is easy to obtain

$$\bar{B}_y(k_x, k_y, z_A, \omega') = -\mathrm{i}B_* b_x \theta(k_x) \times \exp\left(-\frac{\mathrm{i}}{3}k_x^3 b_x^3 - \varepsilon k_x b_x\right)\delta(k_y)\delta(\omega - \omega').$$

On substituting this expression into formula (63a) of our previous paper and by performing an inverse Fourier-transform, we get

$$B_x(x,y,0,t) = B_*(x,y)\,e^{-i\omega t}\int_0^\infty dz\,\frac{\sigma_H(z)}{\Sigma_P} \times\phi\left[\frac{x-(z_A-z)\tan\chi}{b_x}+i\left(\varepsilon+\frac{z}{b_x}\right)\right]. \quad (9)$$

This relationship, combined with (3) and (8), solves the problem of the coupling between the toroidal monochromatic wave in the magnetosphere and the field on the ground induced by it.

Let us note the equality useful for the following treatment which follows from (7), (9) and formula (70a) of our previous paper:

$$\int_{-\infty}^{\infty} dx' P(x-x')\phi\left(\frac{x'}{b_x}+i\varepsilon\right) = \int_0^\infty dz\,\frac{\sigma_H(z)}{\Sigma_P} \times\phi\left[\frac{x-(z-z_A)\tan\chi}{b_x}+i\left(\varepsilon+\frac{z}{b_x}\right)\right]. \quad (10)$$

Let us consider some particular cases of formula (9). Note, at first, that the integral over z is actually taken in the interval $(H, H+\Delta)$, where the function $\sigma_H(z)$ is non-zero. The Hall layer thickness $\Delta = 30$–40 km is significantly less than the height, at which it is located $H \approx 100$ km, and for approximate calculations one can put $H \gg \Delta$. If in this case the inequality

$$b_x \gg \Delta\tan\chi \quad (11)$$

holds then, when integrating over z, the variation of the argument of the function ϕ can be neglected so that

$$B_x(x,y,0,t) = B_*(x,y)\frac{\Sigma_H}{\Sigma_P} \times\phi\left[\frac{x-x_A}{b_x}+i\left(\varepsilon+\frac{H}{b_x}\right)\right]e^{-i\omega t}, \quad (12)$$

where $x_A = (z_A - H)\tan\chi$. If $\varepsilon \lesssim 1$ and H/b_x is also not too large, the structure of the running wave that exists in the magnetosphere must also manifest itself clearly on the Earth's surface. If, however,

$$b_x \ll H, \quad (13)$$

then using the asymptotic representation (6b), from (12) we obtain

$$B_x(x,y,0,t) = B_*\frac{\Sigma_H}{\Sigma_P}\frac{b_x}{x-x_A+i(H+\varepsilon b_x)}e^{-i\omega t}. \quad (14)$$

This same expression, provided that the inequality (13) is satisfied, follows directly from formula (9). In this case the structure of the running wave, even though it exists in the magnetosphere, does not manifest itself on the ground.

When $\varepsilon \gg 1$, the transverse wave structure is also absent in the magnetosphere. In this case, from (7) it follows that

$$B_y(x,y,z_A,t) = B_I\frac{\delta_x}{\delta_x - ix}e^{-i\omega t}, \quad (15)$$

where $B_I = -iB_*/\varepsilon$ is a typical value of the field on the ionosphere, and $\delta_x = \varepsilon\delta_x \equiv (\gamma_N/\Omega_N)(l_N\sqrt{g\dagger}/\cos\chi)$ is a typical scale of its variation along axis x. From (14) we then get

$$B_x(x,y,0,t) = B_I\frac{\Sigma_H}{\Sigma_P}\frac{\delta_x}{H+\delta_x-i(x-x_A)}e^{-i\omega t}. \quad (16)$$

Relationships such as (15) and (16), which are valid when the transverse dispersion of an Alfvén wave has no effect at all, were obtained by Hughes and Southwood as far back as 1976.

Bearing in mind the above values of the parameters, it can be concluded that case (13) most probably occurs when formula (14) or (16) is valid; and only at high latitudes where the parameter b_x can reach values as large as 30 km and the angle χ is small, can case (11) be realized, when the running wave structure, though in a smoothed form, manifests itself on the ground (Fig. 2).

The formulae of the present section permit us to associate oscillation amplitudes of the ground magnetic field with those at any point of the magnetosphere, in particular at the equator where measurements on spacecraft (geostationary satellites, for example) are mostly made. We confine our attention to order-of-magnitude estimates. From (3) follows an estimate of a typical value of the field at the equator

$$B_0 \sim M\hat{H}_N(0)\phi\left(i\frac{\delta_x}{b_x}\right),$$

and from(12) follows that of the field on the ground

$$B_g \sim \frac{\Sigma_H}{\Sigma_P}M\hat{H}_N(x_*^3)\phi\left(i\frac{\delta_x+H}{b_x}\right).$$

From this we have

$$\frac{B_g}{B_0} \sim \frac{\Sigma_H}{\Sigma_P}\frac{\hat{H}_N(x_*^3)}{\hat{H}_N(0)}\frac{\phi[i(\delta_x+H)/b_x]}{\phi[i\delta_x/b_x]}.$$

From formula (4) we have

$$\frac{\hat{H}_N(x_*^3)}{\hat{H}_N(0)} = \left(\frac{A_0}{A_*}\right)^{1/2}\left(\frac{\Delta s_0}{\Delta s_*}\right)^{1/2},$$

where $\Delta s = (g_1 g_2)^{1/2}$ is the cross-sectional area of a thin flux tube having a unit size in coordinates x^1 and

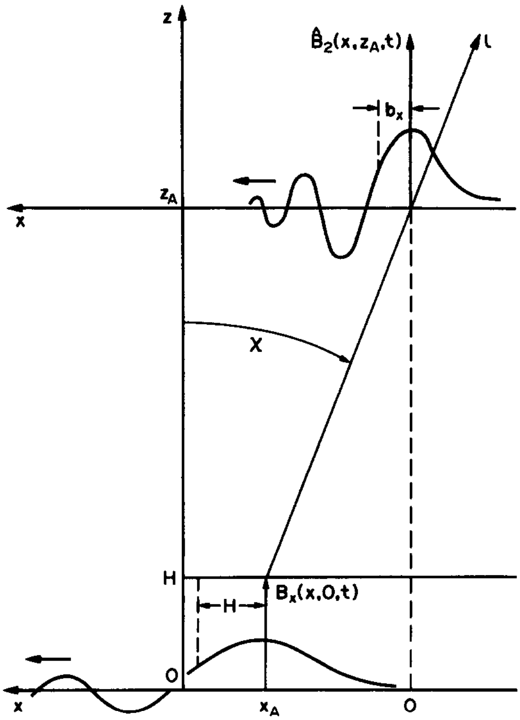

FIG. 2. SCHEME OF PASSAGE OF THE FIELD OF A TOROIDAL ALFVÉN WAVE ($\hat{B}_2$), EXCITED BY MONOCHROMATIC MAGNETOSOUND, FROM THE MAGNETOSPHERE ($z \geqslant z_A$) TO THE EARTH ($z = 0$).
In the mangetosphere, Alfvén oscillations are a standing (between the magneto-conjugate ionospheres) and running (across the magnetic shells) wave; the direction of transverse propagation is indicated by the heavy arrow. The field of the oscillation, induced on the Earth's surface (b_x) also is a running (in the latitudinal direction) wave. It is evident that the oscillation field transport occurs along the field line (axis l). While in the magnetosphere the transverse wavelength b_x is much less than the atmospheric thickness H, on the Earth's surface the typical spatial scale of the oscillation field reaches the value of H.

x^2. Subscripts 0 and * refer to the quantities at the equator and on the upper edge of the ionosphere, respectively. As far as the relation of the function ϕ is concerned, there exist three substantially different cases. We give here the result for each of them:

$$\frac{B_g}{B_0} \sim \frac{\Sigma_H}{\Sigma_P}\left(\frac{A_0}{A_*}\right)^{1/2}\left(\frac{\Delta s_0}{\Delta s_*}\right)^{1/2}$$

$$\times \begin{cases} 1 & \text{when } b_x \gtrsim \delta_x, \quad b_x \gtrsim H, \\ b_x/H & \text{when } b_x \gtrsim \delta_x, \quad b_x \ll H, \\ \delta_x/H, & \text{when } b_x \ll \delta_x, \quad b_x \ll H. \end{cases}$$

The wide range of variation of the parameters involved here leads to the fact that the ratio B_g/B_0 is able to assume, in different cases, values both larger and smaller than unity.

4. THE GROUND FIELD OF A STANDING ALFVEN WAVE EXCITED BY A SUDDEN IMPULSE OF MAGNETIC SOUND

A sudden impulse of magnetic sound having the form of a δ-function of time, is a theoretical idealization which is, in a sense, opposite to a monochromatic oscillation. It might be expected that it models some important features of magnetosonic oscillations excited by a short-duration source such as a substorm explosion or an SSC event. Leonovich and Mazur (1989b) obtained a formula that represents an Alfvén wave excited by such a magnetic sound

$$\hat{B}_2(x^1, x^2, x^3, t) = M(x^1, x^2)\hat{H}_N(x^3)\theta(t) \times e^{-\gamma_N t} \sin[\Omega_N(x^1)t + t^3/6\tau_N^3]. \quad (17)$$

Here $\tau_N = (l_N/\rho_N)^{2/3}\bar{\Omega}_N^{-1}$ is the dispersion time. This wave is also a toroidal one. The above formula refers to a region of monotonic variation of the function $\Omega_N(x^1)$ where expansion (2) is applicable; in this case, however, $\bar{x}^1$ is an arbitrarily chosen surface rather than the resonance shell (it is absent altogether). It is convenient to introduce the designation

$$\omega_N(x) \equiv \Omega_N(x^1) = \Omega_N\left(\bar{x}^1 + \frac{\cos\chi}{\sqrt{g_1^*}}x\right) = \bar{\Omega}_N\left(1 - \frac{x}{l_x}\right),$$

where

$$l_x = l_N\sqrt{g_1^*}/\cos\chi.$$

From the relationships (1) and (17) it follows that on the lower edge of the magnetosphere

$$B_y(x, y, z_A, t) = B_*(x, y)\theta(t) \times e^{-\gamma_N t}\sin[\omega_N(x)t + t^3/6\tau_N^3],$$

where, as before,

$$B_* = M(x^1, x^2)\hat{H}_N(x^3{}_*).$$

Let us Fourier-transform this expression in coordinates x and y (but not in time). Neglecting the weak dependence of M on the coordinates, we have

$$\bar{B}_y(k_x, k_y, t) = \frac{i}{2}B_*\theta(t)\,e^{-\gamma_N t} \times [\exp(-i\bar{\Omega}_N t - it^3/6\tau_N^3)\delta(k_x - q(t)) - \exp(i\bar{\Omega}_N t + it^3/6\tau_N^3)\delta(k_x + q(t))]\delta(k_y). \quad (18)$$

Here it is designated

$$q(t) = \bar{\Omega}_N t/l_x.$$

Expression (18) has a simple physical meaning, namely that as a consequence of the transverse inhomogeneity of the magnetosphere, the spatial structure of the Alfvén wave becomes finer in scale in the transverse direction. In other words, the wave vector k_x increases in accordance with the equation

$$\frac{dk_x}{dt} = -\frac{\partial \omega_N}{\partial x}.$$

On substituting expression (18) into formula (63a) of our previous paper and by performing an inverse Fourier-transform, we obtain

$$B_x(x, y, 0, t) = B_*(x, y)\theta(t)\,e^{-\gamma_N t} \times \int_0^\infty dz \frac{\sigma_H(z)}{\Sigma_P} e^{-q(t)z} \sin[\omega_N(x-(z_A-z)\tan\chi)t + t^3/6\tau_N^3]. \quad (19)$$

Let us introduce the function

$$\bar{\Sigma}_H(q) = \int_0^\infty dz \sigma_H(z) \exp[-q(1-i\tan\chi)(z-H)].$$

When $q\Delta \ll 1$, we have $\bar{\Sigma}_H(q) = \Sigma_H$. When $q\Delta \gg 1$, the function $\bar{\Sigma}_H(q)$ decreases as a power-law, and the exponent of a power depends on the smoothness of the function $\sigma_H(z)$. Using this function the expression (19) can be rewritten as

$$B_x(x, y, 0, t) = B_*(x, y)\frac{|\bar{\Sigma}_H(q(t))|}{\Sigma_P}\theta(t) \times \exp[-(\gamma_N + \bar{\Omega}_N H/l_x)t] \times \sin[\omega_N(x-x_A)t + t^3/6\tau_N^3 - \arg\bar{\Sigma}_H(q(t))].$$

A most important property of the obtained solution is that the spatial structure of the oscillation field gets finer in scale in coordinate x, which is caused by a corresponding phenomenon in the magnetosphere. This results in the addition of $\bar{\Omega}_N(H/l_x)$ to the damping decrement. The implication of the additional damping is quite clear, i.e. as the wave vector k_x increases, the passage of the oscillation field through the atmosphere to the Earth gets more complicated. A typical value of $H/l_x = 0.1$–1, i.e. the damping decrement is sufficiently large. Possibly, this accounts for the rapid damping of the Pi2 trains and for oscillations accompanying SSC. When $q\Delta > 1$, i.e. when $t > (l_x/\Delta)\bar{\Omega}_N^{-1}$, a power-law damping associated with the decrease of the function $\bar{\Sigma}_H(q(t))$, is also added.

The dispersion term $t^3/6\tau_N^3$ begins to play a role when $t \sim \tau_N$; but in this case $\gamma_N t \sim \bar{\Omega}_N(H/l_x)\tau_N = H/b_x$. In typical cases $H \gg b_x$, i.e. the oscillation is damped earlier than the dispersion effect begins to manifest itself. Only at high latitudes can b_x reach values of the order of H and the dispersion must then manifest itself in an increase, with the time, of the observed oscillation frequency. The main features of the above picture are illustrated by Fig. 3.

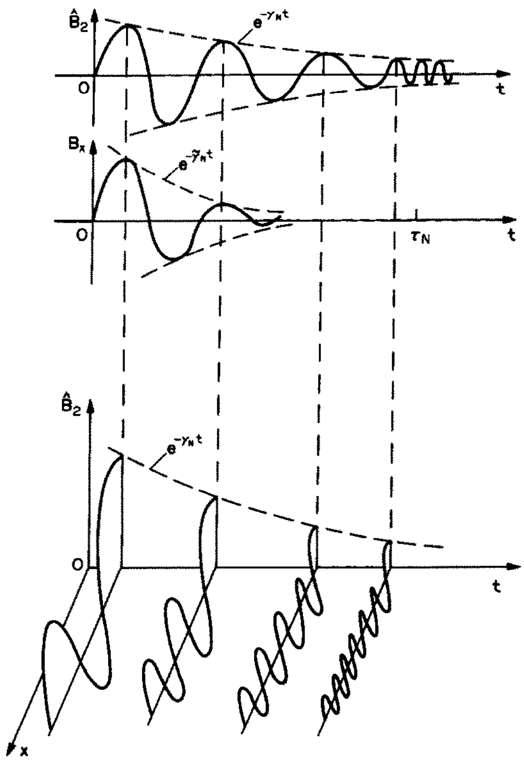

FIG. 3. THE UPPER PART SHOWS THE TIME BEHAVIOUR OF THE ALFVÉN OSCILLATION FIELD IN THE MAGNETOSPHERE [$\hat{B}_2(t)$], EXCITED BY A SUDDEN IMPULSE OF MAGNETOSOUND, AND OF MAGNETIC FIELD OSCILLATIONS INDUCED BY THEM ON THE EARTH'S SURFACE [$B_x(t)$].
The oscillations on the ground are damped considerably more rapidly compared with the oscillations in the magnetosphere (the relationship of the typical damping decrements is $\tilde{\gamma}_N = \Omega_N H/l_x \gg \gamma_N$). This is attributable to the fact that the spatial structure of the oscillations in the magnetosphere get finer because of the presence of a transverse plasma inhomogeneity as shown in the lower part. Similarly, the transverse structure of the oscillations get finer also on the ground but with a significantly larger decrease of the amplitude. The typical dispersion time τ_N denotes the time interval, during which the frequency of Alfvén oscillations varies appreciably.

5. THE GROUND FIELD OF STANDING ALFVEN WAVES EXCITED BY A STOCHASTIC MAGNETIC SOUND

In the two preceding sections, when writing the formulae for the Alfvén wave field [relationships (3) and (17)], we have limited ourselves to considering one longitudinal harmonic. This is quite admissible because a total field is a superposition of fields of different harmonics and each term of the sum can be studied separately. Moreover, in the case of monochromatic oscillations near a given resonance surface, a field of one harmonic only is different from zero; but when considering stochastic oscillations, the problem is that of calculating different correlators, i.e. quadratic combinations of fields; it is necessary then to consider the entire sum.

According to Leonovich and Mazur (1989a),

$$\begin{aligned}\tilde{B}_2(x^1,x^2,x^3,\omega) &= \sum_N \tilde{\mu}_N(x^1,x^2,\omega)\hat{H}_N(x^1,x^3)\tilde{Q}_N(x^1,\omega),\\ \tilde{Q}_N(x^1,\omega) &= \frac{1}{2\alpha_N}\phi\left(\frac{x^1-\bar{x}^1}{\alpha_N l_N}+\frac{\omega-\bar{\Omega}_N+\mathrm{i}\gamma_N}{\alpha_N\bar{\Omega}_N}\right),\\ \alpha_N &= 2^{-1/3}\left(\frac{\rho_N}{l_N}\right)^{2/3},\\ \tilde{\mu}_N(x^1,x^2,\omega) &= \oint \mathrm{e}_N(x^1,x^3)\frac{\partial\tilde{B}_3(x^1,x^2,x^3,\omega)}{\partial x^2}\,\mathrm{d}x^3.\end{aligned} \tag{20}$$

These formulae are also written for a region of monotonic variation of the function $\Omega_N(x^1)$. Here $\mathrm{e}_N(x^1,x^2)$ is a certain function, the expression for which is given in the paper just cited. The field of a magnetosonic wave is a random function. For our purposes, an ensemble of these random functions is defined by the correlator (Leonovich and Mazur, 1989b)

$$\begin{aligned}&\langle\tilde{B}_3^*(x^1,x^2,x^3,\omega)\tilde{B}_3(x^{1\prime},x^{2\prime},x^{3\prime},\omega')\rangle\\ &\quad= b^*(x^1,x^2,x^3,\omega)b(x^{1\prime},x^{2\prime},x^{3\prime},\omega)\delta(\omega-\omega').\end{aligned} \tag{21}$$

The function $b(x^1,x^2,x^3,\omega)$ satisfies the equation for a monochromatic magnetosonic wave. The quantity $|b(x^1,x^2,x^3,\omega)|^2$ at fixed values of x^1, x^2, and x^3 can be treated as the spectral density of magnetic sound at a given point or—at a fixed value of ω—as the distribution of the given spectral harmonic in space.

From (20) it follows that on the lower edge of the magnetosphere

$$\tilde{B}_y(x,y,z_\mathrm{A},\omega) = \sum_N \tilde{\mu}_N(x^1,x^2,\omega)\hat{H}_N(x^1,x^3_*)\tilde{q}_N(x,\omega),$$

where

$$\begin{aligned}\tilde{q}_N(x,\omega) &= \tilde{Q}_N\left(\bar{x}^1+\frac{\cos\omega}{\sqrt{g_1^*}}x,\omega\right)\\ &= \frac{1}{2\alpha_N}\phi\left(\frac{x}{b_x}+\frac{\omega-\bar{\Omega}_N}{\alpha_N\bar{\Omega}_N}+\mathrm{i}\frac{\gamma_N}{\alpha_N\bar{\Omega}_N}\right).\end{aligned}$$

According to formulae (70a) of our preceding paper and to (10) of this paper, on the ground we have

$$\tilde{B}_x(x,y,0,\omega) = \sum_N \tilde{\mu}_N(x^1,x^2,\omega)\hat{H}_N(x^1,x^3_*)\tilde{P}_N(x,\omega),$$

where

$$\begin{aligned}\tilde{P}_N(x,\omega) &= \int_{-\infty}^{\infty}\mathrm{d}x' P(x-x')\tilde{q}_N(x',\omega)\cos\chi\\ &= \frac{\cos\chi}{2\alpha_N}\int_0^\infty \mathrm{d}z\frac{\sigma_\mathrm{H}(z)}{\Sigma_\mathrm{P}}\phi\left[\frac{x-(z_\mathrm{A}-z)\tan\chi}{b_x}\right.\\ &\quad\left.+\frac{\omega-\bar{\Omega}_N}{\alpha_N\bar{\Omega}_N}+\mathrm{i}\left(\frac{\gamma_N}{\alpha_N\bar{\Omega}_N}+\frac{z}{b_x}\right)\right].\end{aligned} \tag{22}$$

The quantity $\tilde{P}_N(x,\omega)$, which is considered to be a function ω at a fixed value of x, is concentrated in the vicinity of $\omega=\omega_N(x-x_\mathrm{A})$ on a scale

$$\Delta\omega \sim \begin{cases}\alpha_N\bar{\Omega}_N & \text{when}\quad \alpha_N > \gamma_N/\bar{\Omega}_N+H/l_x,\\ \gamma_N+(H/l_x)\bar{\Omega}_N & \text{when}\quad \alpha_N < \gamma_N/\bar{\Omega}_N+H/l_x.\end{cases} \tag{23}$$

In the first case the function $\tilde{P}_N(x,\omega)$ has the character of the oscillation in ω with a "wavelength" $\alpha_N\bar{\Omega}_N$.

The field $\tilde{B}_x(x,y,0,\omega)$ is a random function, whose properties are defined by different correlators. Let us calculate

$$\begin{aligned}&\langle\tilde{B}_x^*(x,y,0,\omega)\tilde{B}_x(x,y,0,\omega')\rangle\\ &\quad= \sum_{N,N'}\langle\tilde{\mu}_N^*(x^1,x^2,\omega)\tilde{\mu}_{N'}(x^1,x^2,\omega')\rangle\hat{H}_N(x^1,x^3_*)\\ &\qquad\times\hat{H}_{N'}(x^1,x^3_*)\tilde{P}_N^*(x,\omega)\tilde{P}_{N'}(x,\omega').\end{aligned}$$

In this sum it is possible to retain only terms with $N=N'$ because, when $N\neq N'$, the product $\tilde{P}_N^*\tilde{P}_{N'}$ is virtually zero, i.e. monochromatic Alfvén oscillations corresponding to two different resonance surfaces do not overlap one another. Besides, from (21) it follows that

$$\langle\tilde{\mu}_N^*(x^1,x^2,\omega)\tilde{\mu}_N(x^1,x^2,\omega')\rangle = |\tilde{\beta}_N(x^1,x^2,\omega)|^2\delta(\omega-\omega'),$$

where

$$\tilde{\beta}_N(x^1,x^2,\omega) = \oint \mathrm{e}_N(x^1,x^3)\frac{\partial\tilde{b}(x^1,x^2,x^2,\omega)}{\partial x^2}\,\mathrm{d}x^3.$$

As a result, we obtain

$$\langle \tilde{B}_x^*(x, y, 0. \omega)\tilde{B}_x(x, y, 0, \omega')\rangle = m(x, y, \omega)\delta(\omega-\omega').$$

Here it is designated

$$m(x, y, \omega) = \sum_N |\tilde{\beta}_N(x^1, x^2, \omega)|^2 \times \hat{H}_N^2(x^1, x_*^3)|\tilde{P}_N(x, \omega)|^2.$$

This function can be regarded as the spectral density of oscillations at a given point of the terrestrial surface. The presence of the terms $|\tilde{P}_N|^2$ means that the spectrum consists of a sequence of peaks at frequencies $\omega = \omega_N(x-x_A)$, whose width is defined by relationship (23). For typical values of the parameters $\Delta\omega \sim (H/l_x)\bar{\Omega}_N = (0.1\text{–}1)\bar{\Omega}_N$.

It is also of interest to consider a mean square of the pulsation amplitude on the ground. We have

$$\langle B_x^2(x, y, 0, t)\rangle = 2\int_0^\infty m(x, y, \omega)\,d\omega$$

$$\approx 2\sum_N |\tilde{\beta}_N(x^1, x^2, \Omega_N(x^1))|^2 \hat{H}_N^2(x^1, x_*^3) \times \int_0^\infty |\tilde{P}_N(x, \omega)|^2\,d\omega.$$

In this equality, by assuming that $|\tilde{P}_N|^2$ is a sharp peak, we factor the smooth function $|\tilde{\beta}_N(x^1, x^2, \omega)|^2$ outside the integral sign over ω, at point $\omega = \omega_N(x-x_A) \approx \Omega_N(x^1)$. Using the expression (22) and the integral representation (5) it is easy to calculate the last integral. As a result, we obtain

$$\langle B_x^2(x, y, 0, t)\rangle = \frac{\pi}{2}\sum_N \Omega_N(x^1) \times |\tilde{\beta}_N(x^1, x^2, \Omega_N(x^1))|^2 \hat{H}_N^2(x^1, x_*^3)\cos^2\chi \times \int_0^\infty dz \frac{\sigma_H(z)}{\Sigma_P}\int_0^\infty dz' \frac{\sigma_H(z')}{\Sigma_P} \times \frac{l_x\left(H+\frac{\gamma_N}{\Omega_N}l_x+\frac{z+z'}{2}\right)}{\left(H+\frac{\gamma_N}{\Omega_N}l_x+\frac{z+z'}{2}\right)^2+\frac{(z-z')^2\tan^2\chi}{4}}.$$

Under the assumption that $\Delta \ll H$ and $\gamma_N/\Omega_N \ll H/l_x$ this expression simplifies considerably:

$$\langle B_x^2(x, y, 0, t)\rangle = \frac{\pi}{2}\left(\frac{\Sigma_H}{\Sigma_P}\right)^2 \frac{l_x\cos^2\chi}{H}\sum_N \Omega_N(x^1) \times |\tilde{\beta}_N(x^1, x^2, \Omega_N(x^1))|^2\ \hat{H}_N^2(x^1, x_*^3). \quad (24)$$

These relationships can be used for interpreting the distribution, over the Earth's surface, of the amplitude of Pc3–Pc5 pulsations. In this regard it is mainly possible to repeat the conclusions drawn in a paper by Leonovich and Mazur (1989b), by taking into account of course the differences of formulae for the field on the ground from those for the field in the magnetosphere. A detailed analysis of this question is beyond the scope of the present paper, and we shall limit ourselves to making one remark. In the plasmapause region a typical scale l_x decreases abruptly. In accordance with formula (24), this might account for the presence of a deep minimum in the meridional profile of the Pc3 amplitude in this region (see Fig. 4).

6. THE GROUND FIELD OF A MONOCHROMATIC STANDING ALFVEN WAVE OF THE POLOIDAL TYPE

The main difference of poloidal Alfvén waves from toroidal ones is that the value of the azimuthal wave vector is large. Hence, their spatial scale along the normal to the magnetic shell turns out to be much larger than the azimuthal scale. To this, there corresponds the radial polarization of a disturbed magnetic field: $\hat{B}_1 \gg \hat{B}_2$. Leonovich and Mazur (1990) have obtained formulae governing the field of a monochromatic Alfvén wave of the poloidal type having a given value of the azimuthal component of the wave vector. They are representable as

$$\hat{B}_1(x^1, x^2, x^3, t) = M\hat{P}_N(x^3)\phi\left(\frac{x^1-\bar{x}^1}{b}+i\varepsilon\right) \times \exp(ik_2x^2 - i\omega t), \quad (25a)$$

$$\hat{B}_2(x^1, x^2, x^3, t) = i\frac{M}{k_2 b}\left(\frac{g_2}{g_1}\right)^{1/2}\hat{P}_N(x^3) \times \phi'\left(\frac{x^1-\bar{x}^1}{b}+i\varepsilon\right)\exp(ik_2x^2 - i\omega t). \quad (25b)$$

Here k_2 is the covariant azimuth component of the wave vector. If the azimuthal angle φ is used as the coordinate x^2, then $k_2 = m$ is the azimuthal wavenumber. The function $\hat{P}_N(x^3)$ represents the longitudinal structure of the mode and is, in its meaning, similar to the function $\hat{H}_N(x^3)$. As a matter of fact, these functions differ, but for $N \gg 1$ the difference disappears so that for $\hat{P}_N(x^3)$, the expression (4) can be used. The function ϕ has the same meaning as before, but the parameters b and ε are different here:

$$b = 2^{-1/3}r_N^{2/3}l_N^{1/3}, \quad \varepsilon = (l_N/b)(\gamma_N/\Omega_N),$$

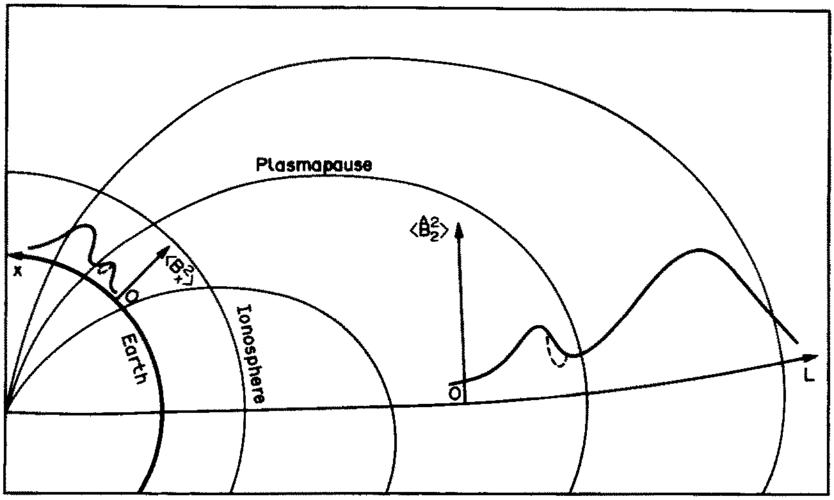

FIG. 4. THE LATITUDE DISTRIBUTION OF THE MEAN SQUARE OF THE AMPLITUDE OF ALFVÉN OSCILLATIONS OF THE MAGNETOSPHERE ($\langle \hat{B}_2^2 \rangle$) EXCITED BY STOCHASTIC MAGNETOSOUND, AND OF THE MAGNETIC FIELD OSCILLATIONS INDUCED BY THEM ON THE EARTH'S SURFACE ($\langle B_x^2 \rangle$).
The dashed line in the plot shows the possible behaviour of the amplitude profile in the region of the dissipative layer if this layer does not coincide with the plasmapause—in this case the formation of a profile with two local minima is possible.

where r_N is the dispersion length of poloidal Alfvén waves, associated with curvature of field lines (for more details see the paper cited above). By the order of magnitude

$$r_N \sim (A_*/A_0)^{1/2}(L/Nm),$$

where L is the field line length, and m is the above-mentioned wave number.

Importantly, for not too large values of m, for $m \lesssim 100$, say, the parameter r_N is much larger than the analogous parameter ρ_N for toroidal waves. This means that the dispersion-induced properties of the wave must be more pronounced for poloidal oscillations as compared with toroidal ones. In particular, for them the parameter $\varepsilon \ll 1$ and we, by neglecting further the dissipation influence, shall assume $\varepsilon = 0$. For the same reason, we examine both transverse components of the disturbed field. Their ratio $\hat{B}_2/\hat{B}_1 \sim (\hat{k}_2\hat{b})^{-1}$, though being small, but not so small as in the toroidal wave.

It is necessary to give one more explanation. Leonovich and Mazur (1990) have considered poloidal eigen-oscillations, but the question of their sources was not treated. Therefore, the equation obtained in their paper for the transverse structure of the oscillation field is a homogeneous one. Its solution is represented in terms of the Airy function $Ai(\xi)$ which is a standing wave, i.e. a superposition of a wave arriving at the resonance surface and of a wave escaping from it; but if the more realistic point of view is adopted and the oscillation source is included in the treatment, then the transverse equation becomes inhomogeneous and its solution will be represented by the function $\phi(\xi)$, i.e. a wave escaping from the resonance surface where it is generated by the source. For that reason, formulae (16) involve the function $\phi(\xi)$.

According to the equality (1), from (25) we have

$$B_n(x, y, z_A, y) = B_*\phi(x/b_x)\exp(ik_y y - i\omega t),$$

$$B_y(x, y, z_A, t) = iB_*(k_y b_x \cos\chi)^{-1} \times \phi'(x/b_x)\exp(ik_y y - i\omega t).$$

Here

$$B_* = MP_N(x_*^1), \quad b_x = b\sqrt{g_1^*}/\cos\chi,$$
$$k_y = k_2/\sqrt{g_2^*}.$$

Corresponding Fourier-transforms have the form

$$\bar{B}_n(k_x', k_y', z_A, \omega') = -iB_* b_x \theta(k_x) \times \exp\left(-\frac{i}{3}k_x'^3 b_x^3\right)\delta(k_y - k_y')\delta(\omega - \omega'),$$

$$\bar{B}_y(k_x', k_y', z_A, \omega') = iB_* b_x \frac{k_x'}{k_y' \cos\chi}\theta(k_x') \times \exp\left(-\frac{i}{3}k_x'^3 b_x^3\right)\delta(k_y - k_y')\delta(\omega - \omega').$$

A typical value of $k_x \sim b_x^{-1}$ is much less than k_y, and this means, in particular, that $k_t = |k_y|$. On substituting the obtained expressions into formulae (63) of our previous paper and by performing, then, an inverse Fourier-transform, we obtain

$$B_x(x,y,0,t) = \mathrm{i}\frac{B_*}{\cos\chi}\frac{1}{k_y b_x}\exp(\mathrm{i}k_y y - \mathrm{i}\omega t)\int_0^\infty \mathrm{d}z$$

$$\times \mathrm{e}^{-|k_y|z}\left[\frac{\sigma_H(z)}{\Sigma_P} - \mathrm{i}\frac{k_y}{|k_y|}\frac{\sigma_P(z)}{\Sigma_P}\right]\phi'\left[\frac{x-(z_A-z)\tan\chi}{b_x}\right],$$

$$B_y(x,y,0,t) = -\frac{B_*}{\cos\chi}\exp(\mathrm{i}k_y y - \mathrm{i}\omega t)\int_0^\infty \mathrm{d}z$$

$$\times \mathrm{e}^{-|k_y|z}\left[\frac{\sigma_H(z)}{\Sigma_P} - \mathrm{i}\frac{k_y}{|k_y|}\frac{\sigma_P(z)}{\Sigma_P}\right]\phi\left[\frac{x-(z_A-z)\tan\chi}{b_x}\right].$$

If the inequalities

$$k_y\Delta \ll 1, \quad \Delta \ll b_x$$

are satisfied, then these relationships are simplified considerably

$$B_x(x,y,0,t) = \mathrm{i}\frac{B_*}{\cos\chi}\frac{1}{k_y b_x}\left(\frac{\Sigma_H}{\Sigma_P} - \mathrm{i}\frac{k_y}{|k_y|}\right)$$

$$\times \mathrm{e}^{-|k_y|H}\phi'\left(\frac{x-x_A}{b_x}\right)\exp(\mathrm{i}k_y y - \mathrm{i}\omega t),$$

$$B_y(x,y,0,t) = -\frac{B_*}{\cos\chi}\left(\frac{\Sigma_H}{\Sigma_P} - \mathrm{i}\frac{k_y}{|k_y|}\right)$$

$$\times \mathrm{e}^{-|k_y|H}\phi\left(\frac{x-x_A}{b_x}\right)\exp(\mathrm{i}k_y y - \mathrm{i}\omega t).$$

From the expressions obtained it is evident that, for poloidal oscillations, the running wave structure, defined by the functions ϕ and ϕ', must manifest itself quite clearly on the ground. The wave's polarization in the terrestrial plane is an elliptic one, with the major axis directed approximately along the parallel. The ratio of semi-axes $B_x/B_y \sim (k_y b_x)^{-1}$ is a small, but quite perceptible, value (see Fig. 5).

7. CONCLUSIONS

The examples considered in the present paper apply to procedures and methods. They possibly reflect some substantial properties of geomagnetic pulsations but cannot be regarded as giving a sufficiently adequate interpretation of some or other kinds of them. This requires that suitable investigations be carried out, with the use of an adequate model of the oscillation source and with the purpose of making a detailed comparison of the theory with observational data and, perhaps, of conducting new problem-oriented experiments. It is our hope that the theory, developed in our papers, of standing Alfvén waves in an axisymmetric magnetosphere and of their penetration to the Earth is capable of forming the basis for such investigations.

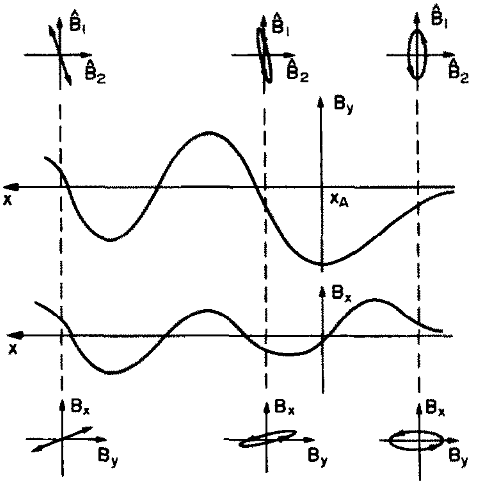

FIG. 5. THE MIDDLE SHOWS THE LATITUDINAL STRUCTURE OF TWO HORIZONTAL COMPONENTS OF MAGNETIC FIELD OSCILLATIONS INDUCED ON THE EARTH'S SURFACE BY POLOIDAL ALFVÉN OSCILLATIONS OF THE MAGNETOSPHERE.
The upper part gives the hodographs of a disturbed magnetic field of poloidal oscillations of the magnetosphere, corresponding to different points on the plots $B_x(x)$ and $B_y(x)$. The lower part gives the respective hodographs of magnetic field oscillations on the Earth's surface. There is a rotation of the ellipse of polarization of the oscillations by $\pi/2$ between the magnetosphere and the Earth associated with the influence of ionospheric Hall conductivity. Besides, it is evident that the orientation and the shape of the ellipse depend on the point of observation.

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