Linear transformation of the standing Alfven wave in an axisymmetric magnetosphere

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Abstract. In an attempt to further develop the theory of transversally small-scale standing Alfven waves, constructed by Leonovich and Mazur (Planet. Space Sci. 41, 697–717, 1993) their spatial structures are investigated in the neighbourhood of the toroidal resonance surface. Near this surface, of important significance are the effects of finite Larmor radius of ions and of electron inertia which lead to the so-called kinetic dispersion of Alfven waves. It is shown that near the toroidal surface there occurs a total linear transformation of one kind of standing Alfven waves, whose dispersion is due to the curvature of geomagnetic field lines (these were thoroughly investigated in the above-cited paper), into standing Alfven waves of another type, kinetic waves. Formulas have been obtained which thoroughly define the spatial structure of the waves under consideration.

1. Introduction

This paper is a direct continuation of our earlier publication (Leonovich and Mazur, 1993) devoted to the theory of transversally small-scale standing Alfven waves in the axisymmetric magnetosphere. It should here be emphasized that in that paper the small-scale character of the waves was assumed not only in the direction normal to the magnetic shells (which is quite natural, by virtue of the inhomogeneity of plasma in this direction) but also in the azimuthal direction. In other words, waves with large azimuthal wave numbers, \( m \gg 1 \), were considered.

The theory was developed within the framework of ideal magnetic hydrodynamics. In this approximation in a homogeneous plasma and in a homogeneous magnetic field, Alfven waves are known to be devoid of transverse dispersion. But Leonovich and Mazur (1990) showed that in a magnetic field with curved field lines, transversally small-scale Alfven waves involve a specific transverse dispersion which leads to a slow (as compared with the Alfven velocity) displacement of standing Alfven waves across the magnetic shells. Consequences of this effect were studied in detail by Leonovich and Mazur (1993). They showed that a standing Alfven wave is generated by external sources near a given magnetic shell where its polarization has a poloidal character (the electric field oscillates azimuthally, and the magnetic field oscillates in the direction normal to the magnetic shell), displaces then to another shell where it has toroidal character (the magnetic field oscillates azimuthally, and the electric field oscillates in the direction normal to the magnetic shell), and is totally absorbed on this shell. We termed the first and second shells, respectively, “poloidal and toroidal resonance surfaces”.

In our developed theory we used the WKB approximation in a coordinate normal to the magnetic shells. It turns out that the corresponding quasi-classical wave vector is zero on the poloidal surface and becomes infinite on the toroidal surface. Note that this fully agrees with the polarization of the wave. The extreme comminution of the transverse spatial structure of the field wave, as the wave approaches the toroidal surface, leads to two important consequences. On the one hand, there is a decrease of the contribution of the dispersion associated with the curvature of the field lines, which is manifested in that the transverse group velocity, caused by this dispersion, tends rapidly to zero. On the other hand, there is an increase of the role of the better known dispersion of the Alfven waves, caused by effects which are beyond the scope of an ideal MHD, namely the inertia of the electrons and the finite Larmor radius of the ions. Alfven waves, for which such a dispersion is important, received the name of kinetic waves (see, for example, Hasagawa and Uberoi (1982)). The combination of these two factors leads to the need to take into account the kinetic dispersion when investigating the spatial structure of a standing Alfven wave near the toroidal surface. This is the subject of the present study.
Kinetic Alfven waves have already been addressed in numerous publications on hydrodynamical oscillations of the magnetosphere (Hasegawa, 1976; see also reviews by Southwood and Hughes, 1983; Goertz, 1984). However, those publications used a magnetospheric model in the form of a plane plasma sheet (a homogeneous magnetic field and a one-dimensionally inhomogeneous (in the transverse direction) plasma). Kinetic Alfven waves in the magnetospheric model with curved geomagnetic field lines and an inhomogeneous (in both the transverse and longitudinal directions) plasma were considered by Leonovich and Mazur (1989). They investigated the resonant excitation of a standing Alfven wave by monochromatic magnetosound penetrating into the magnetosphere from outside. Their treatment involved examining oscillations with small values of azimuthal wave number, \( n \sim 1 \), because only for such \( n \) magnetosound can penetrate deep into the magnetosphere. It was shown that magnetosound excites a kinetic Alfven wave in the neighbourhood of the toroidal surface, and from this surface the wave moves slowly away and is damped gradually due to the Ohmic dissipation at ionospheric terminations. This result differs substantially from the pattern of the event when \( n \gg 1 \). In this last case, as will be shown later, the Alfven wave that is generated far from the toroidal surface, undergoes—in its immediate vicinity—a linear transformation to the kinetic Alfven wave.

2. The derivation of the equation for the spatial structure of the Alfven wave

In this paper we will use the notions and notations introduced by Leonovich and Mazur (1993). In particular, the axisymmetrical magnetosphere will be described in terms of an orthogonal curvilinear coordinate system \( (x', s', s_3) \), in which the coordinate \( s' \) characterizes the magnetic shell, the coordinate \( s_3 \) represents the field line on this shell (the azimuthal angle \( \varphi \) can be used as \( s' \)), and the coordinate \( x' \) varies along the field line. The diagonal components of the metric tensor will be denoted by \( g_{11}, g_{22}, g_{33} \), and \( g = g_{12}g_{22}g_{33} \) is its determinant.

The disturbed electric field of a monochromatic wave obeys the equation

\[
\nabla \times \nabla \times \mathbf{E} = \frac{\omega^2}{c^2} \mathbf{E}
\]

where \( \nabla \times \) is the dielectric constant tensor. The hydromagnetic oscillations of interest are low-frequency ones, \( \omega \ll \omega_c \), where \( \omega_c = eB/m_e c \) is the gyrorfrequency of the ions, and are relatively large-scale ones, \( k_z \rho_i \ll 1 \), where \( k_z \) is the physical meaning of the transverse wave vector, \( \rho_i \) being the Larmor radius of the ions. For such oscillations, the physical components (i.e. the components in a local Euclidean basis) of the dielectric constant tensor have the form (Akhiezer et al., 1974)

\[
\begin{align*}
\hat{\varepsilon}_{11} &= \frac{c^2}{A^2} (1 - \frac{3}{4} k_z^2 \rho_i^2), \\
\hat{\varepsilon}_{22} &= \frac{c^2}{A^2} (1 - \frac{3}{4} k_z^2 \rho_i^2). 
\end{align*}
\]

Here \( A = B_0 \sqrt{4 \pi n_e} \) is the Alfven velocity, \( \hat{k}_1 \) and \( \hat{k}_2 \) are the physical components of the wave vector

\[
\begin{align*}
\hat{\Lambda}_2 &= -\frac{\rho_i^2}{n}\left[ k_z \nabla \times \nabla \times \mathbf{E} \right]_1, \\
\hat{\rho}_i &= \frac{v_e}{\omega}, \\
\hat{\rho}_i &= \frac{v_i}{\omega}, \\
\hat{\nu}_z &= \left( \frac{T_z}{m_e} \right)^{1/2}, \\
\hat{\nu}_x &= \left( \frac{T_x}{m_i} \right)^{1/2}, \quad \hat{\nu}_y = \left( \frac{T_y}{m_i} \right)^{1/2}
\end{align*}
\]

and \( w(z) \) is a function well known in plasma physics (see Fried and Conte, 1961)

\[
w(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^\pi \frac{t \exp(-t^2/2)}{t - z} \, dt = 1 - z e^{-z^2/2} \int_0^\infty e^{-t^2} \, dt + i \left( \frac{\pi}{2} > 1 \right) z e^{-z^2/2}.
\]

For real \( z \), this function has the following limiting expressions:

\[
w(z) = \begin{cases} 
1 + i(\pi/2)^{1/2}z, & |z| \ll 1 \\
1 + i \left( \frac{\pi}{2} \right)^{1/2} z e^{-z^2/2}, & |z| \gg 1
\end{cases}
\]

In the approximation of an ideal MHD \( \rho_i = \Lambda_2 = 0 \). In this case relationships (1) and (2) describe, in a homogeneous plasma, the independent Alfven and magnetosound waves with the dispersion laws \( \omega^2 = \frac{k_z^2 A^2}{1 + k_z^2 \Lambda_2} \) and \( \omega^2 = \frac{k_z^2 A^2}{1 + k_z^2 \Lambda_2} \), respectively. A specific property of the Alfven waves is the absence of the transverse dispersion: the frequency \( \omega \) does not depend on \( k_z \). When going beyond the scope of an ideal MHD, they are imparted a weak transverse dispersion. In a homogeneous plasma, from (1) and (2) it follows that

\[
\omega^2 = k_z^2 A^2 (1 + k_z^2 \Lambda_2), \quad \Lambda_2 = \Lambda_2^2 + \frac{1}{2} \rho_i^2.
\]

Such Alfven waves received the name kinetic waves. The corresponding dispersion will also be referred to as the kinetic dispersion here.

Even in the presence of a dispersion, the approximate equality \( \omega \approx k_z A \) is valid, and in the argument of the function \( w \) one may put

\[
\frac{\omega}{k_z \nu_e} = \frac{A}{\nu_e} \equiv \frac{s \beta_e}{(m_i/m_e)}
\]

where \( s = c/\omega_m \) is the electron skin length, and \( \beta_e = 8 \pi n_e T_e / B_0^2 \) the ratio of electron to magnetic pressure. Hence

\[
\Lambda_2 = \frac{\rho_i^2}{w(s/\beta_e)}.
\]

In particular
When $s \ll \rho_i (i.e. \beta_i \sim m_i/m_e)$, the quantity $\Lambda_i^2$ is a complex one, and $\text{Im} \Lambda_i^2 < 0$. From (4) it follows that this corresponds to the damping of the wave. It is caused by the Cherenkov resonance due to electrons, which is effective by virtue of the relationship $s \sim A$. In the magnetospheric plasma the values of $s$ and $\rho_i$ are extremely small (varying from a few hundred meters to several tens of kilometers) compared with typical scales of the magnetosphere. Therefore, the kinetic dispersion has a role only for extremely small-scale waves in the transverse direction.

In an inhomogeneous plasma and in an inhomogeneous magnetic field, relationships (1) and (2) can be brought to a system of differential equations for covariant components of the electric field of the wave $E_i$ ($i = 1, 2, 3$). These are related to the physical components by the equality $E = \frac{1}{\sqrt{g}} E_i g_i$. The quantities $k_i$ in the expression for the dielectric constant tensor should be treated as the operators $k_i = -i(\nabla_i \frac{1}{\sqrt{g}}) \frac{1}{\sqrt{g}}$. Since the terms that contain the operators $k_i$ play an important role in waves of extremely small scale, it may be assumed that they commute with functions that describe equilibrium parameters of the plasma and the magnetic field (i.e., the derivatives $\nabla_i$ in these operators can be referred only to fields $E_i$).

From (1) and (2) when $k_i^2 \rho_i^2 \ll 1$ it is easy to obtain the expression for the longitudinal component $E_3$:

$$E_3 = \Lambda_3 \left( \frac{1}{g_1} \nabla_1 \nabla_3 E_1 + \frac{1}{g_2} \nabla_2 \nabla_3 E_2 \right).$$

After that, the system of equations for the transverse components $E_1$ and $E_2$ can be reduced to the form

$$(\hat{\mathcal{P}}^i + \hat{L}^i) E_i = 0$$

where the operator $\hat{\mathcal{P}}^i$ has the same form as in Leonovich and Mazur (1993):

$$\hat{\mathcal{P}}^i = \nabla_i \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \frac{1}{\sqrt{g}}, \quad \nabla_i = (\nabla_2, -\nabla_1)$$

and $\hat{L}^i$ differs from that in the previous paper by the presence of dispersion additions:

$$\hat{L}^i = \nabla_i \frac{g_{ij} \omega_j}{\sqrt{g} g_3 A^2} \frac{g_{ij} \omega_j}{\sqrt{g} g_3 A^2} \nabla_i - \nabla_3 \frac{g_{ij} \Lambda_3^2}{g_i} \nabla_3 \nabla_i$$

Equation (8) describes the spatial structure of an Alfvén wave. It differs from an analogous equation reported in Leonovich and Mazur (1993) by the presence of two last terms.

The boundary condition on the ionosphere for the potential $\Phi$ has the same form as in the cited paper:

$$\Phi|_{\ell^+} = \frac{1}{4\pi} \int \frac{\partial^2}{\partial x_i \partial x_j} \frac{\partial^2}{\partial y_i \partial y_j} \left( \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \frac{1}{\sqrt{g}} \frac{\partial}{\partial y_j} \right) \Phi dx dy$$

Here $\ell^+$ represents the coordinates of the ionospheric ends of the field line, $\chi$ refers to angles they make with the local vertical, and $\Sigma_{\ell^+}$ corresponds to integral Pedersen conductivities of the conjugate ionospheres. Dispersion effects in (9) can be neglected because the dispersion parameters $\rho_i$ and $\Lambda_i$ on the ionosphere are much smaller than those in the magnetosphere.

In order to pass from the partial differential equation (8) to an ordinary differential equation that describes the mode structure near the toroidal surface, we avail ourselves of the perturbation theory based on the closeness of the desired solution to the toroidal mode. This means that this solution can be represented as

$$\Phi = [\nu_1, (x') \overline{T}_1 (r', \varphi') + \omega_1] \exp i \gamma x$$

Here $k_2$ is the azimuthal wave vector in $x' = \varphi$ is the azimuthal angle, then $k_2 = m$ is the azimuthal wave number, $T_1$ a toroidal wave function, and $\varphi$, a small correction. The function $T_1$ is the eigen-solution of the longitudinal problem

$$\hat{\mathcal{L}}(\Omega_{T_1}) T_1 = \frac{\partial}{\partial r} \left( \frac{\partial T_1}{\partial r} \right) + \Omega_{T_1}^2 T_1 = 0.$$

Here $\Omega_{T_1} = \Omega_{T_1} (x')$ are toroidal eigenfrequencies. The desired function in (10) is the function $\nu_1 (x')$ that defines
the transverse structure of the mode. The equation describing it is the solvability condition for the correction \( \varphi_x \).

Proceeding along similar lines as in Leonovich and Mazur (1993), we obtain

\[
\omega^2 \Lambda^3 \frac{d^2 V_x}{dx^2} + \frac{d}{dx} \left[ (\omega + i \gamma_x) \frac{d V_x}{dx} - \Omega^2 \frac{d V_x}{dx} \right] - k^2 w_{TX} V_x = 0. \tag{11}
\]

Here

\[
\Lambda^3 = 1 \left[ \frac{1}{g_1} (\Lambda^2 + \tau_0) + A^2 \right] \left[ \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \right] A^2 \left( \frac{\partial}{\partial \varphi} \right) \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} A^2 \left( \frac{\partial}{\partial \varphi} \right) \frac{\partial}{\partial \varphi} A^2 \left( \frac{\partial}{\partial \varphi} \right) \right] P \frac{d \varphi}{d \varphi} d \varphi /
\]

\[
w_{TX} = - \left[ \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \right] A^2 \left( \frac{\partial}{\partial \varphi} \right) \frac{\partial}{\partial \varphi} A^2 \left( \frac{\partial}{\partial \varphi} \right) \frac{\partial}{\partial \varphi} A^2 \left( \frac{\partial}{\partial \varphi} \right) \right] P \frac{d \varphi}{d \varphi} d \varphi /
\]

and \( \gamma_x \) is the decrement of damping of the mode on the ionosphere, and the expression for it is given in the cited paper (it is assumed that \( \gamma_x \ll \omega \)). In that paper it is also emphasized that the value of \( w_{TX} \) is nonzero as a consequence of the curvature of geomagnetic field lines, and it is shown that for realistic magnetospheric models it is positive. The expressions for \( \rho_{\varphi} \) and \( w_{TX} \) simplify considerably for harmonics with \( N \gg 1 \), when the WKB approximation in coordinate \( \varphi \) is applicable. In this case

\[
\Lambda^3 = 1 - \frac{\left( \Omega^2 + \tau_0 \right)}{g_1} \frac{d \varphi}{d \varphi} \frac{d \varphi}{d \varphi} \frac{d \varphi}{d \varphi} \frac{d \varphi}{d \varphi} \right] P \frac{d \varphi}{d \varphi} d \varphi /
\]

\[
w_{TX} = - \left[ \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \right] A^2 \left( \frac{\partial}{\partial \varphi} \right) \frac{\partial}{\partial \varphi} A^2 \left( \frac{\partial}{\partial \varphi} \right) \frac{\partial}{\partial \varphi} A^2 \left( \frac{\partial}{\partial \varphi} \right) \right] P \frac{d \varphi}{d \varphi} d \varphi /
\]

Equation (11) differs from an analogous equation from our previous work by the presence of the first term. An important role is played by the sign of the dispersion parameter \( \Lambda^3 \). In the inner part of the magnetosphere where \( \beta_0 = \mu_0/\mu \) and, consequently, \( \gamma_x \ll \rho_{\varphi} \) and \( \Lambda^3 \approx - \gamma_x^2 \) it is negative. In the outer magnetosphere where \( \beta_0 \gg \mu_0/\mu \) and \( \Lambda^3 \approx \rho_{\varphi}^2 \) it is positive. In the intermediate region this parameter is a complex one, and \( \text{Im} \Lambda^3 < 0 \).

3. Linear transformation of a standing Alfven wave near the toroidal resonance surface

If the WKB approximation is applied to equation (11), that is, the solution is sought in the form

\[
V_x \sim \exp \left( i \frac{x}{x} \frac{\varphi}{\varphi} \right) \tag{13}
\]

then for the quasi-classical wave vector \( k_1 \) we obtain (by neglecting the damping on the ionosphere) the equation

\[
\omega^2 \Lambda^3 k_1 - (\omega^2 - \Omega^2) k_1^2 - k^2 w_{TX} = 0. \tag{14}
\]

Alternatively, this equation may be treated as a relationship that relates the local frequency \( \omega \) to the wave vector \( k_1 \). By solving it for \( \omega \) and taking into consideration that \( |k_1^2 \rho_{\varphi}^2| \ll 1 \), we obtain a local dispersion equation

\[
\omega^2 = \Omega^2 + k_1^2 \Lambda^3 \Omega^2 - k^2 w_{TX}. \tag{15}
\]

The last two terms on the right-hand side represent dispersion corrections. For relatively large-scale waves, such that

\[
k_1^2 \ll \frac{k^2 w_{TX}}{k^2}
\]

one can put

\[
\omega^2 = \Omega^2 - k_1^2 w_{TX}. \tag{17}
\]

These waves were investigated in our previous paper and were referred to as small-scale waves. This implied that their transverse wavelength is much smaller than typical magnetospheric scales (it should be noted that the limit \( k_1^2 \to 0 \) cannot be considered in (17); this would imply violating the applicability condition for the WKB approximation: the relevant criteria were considered in the cited paper). In the present study the waves that satisfy condition (16), will be referred to as large-scale ones by reserving the term of the small-scale wave for the inverse case

\[
k_1^2 \gg \frac{k^2 w_{TX}}{k^2}
\]

In this case

\[
\omega^2 = \Omega^2 + (1 + k_1^2 \Lambda^3). \tag{18}
\]

This dispersion equation is quite similar to equation (4) for kinetic Alfven waves. Based on (18) one can determine the group velocity of a small-scale standing Alfven wave in coordinate \( x^1 \):

\[
v_{g_1}^1 = \frac{2 \omega \Lambda^3}{k_1^2}. \tag{19}
\]

Note that when \( \Lambda^3 > 0 \) (i.e. in a rather hot plasma, \( \beta_0 \gg \mu_0/\mu \)) the signs of group velocity \( v_{g_1}^1 \) and phase velocity \( \omega/k_1 \) coincide, and when \( \Lambda^3 < 0 \) (in a cold plasma, \( \beta_0 \ll \mu_0/\mu \)) they are opposite.

We now return to the usual (for the WKB approximation) treatment of relationship (14) as an equation for \( k_1 \). From it we have

\[
k_1^2 = \left( \frac{- \omega^2 - \Omega^2}{2 \omega^2 \Lambda^3} \right)^2 \pm \sqrt{\left( \frac{- \omega^2 - \Omega^2}{2 \omega^2 \Lambda^3} \right)^2 + 4k^2 w_{TX} \Lambda^3 \Omega^2 \frac{d \varphi}{d \varphi} \frac{d \varphi}{d \varphi}}. \tag{20}
\]

This equality defines the function \( k_1^2 = k_1^2(x^1) \), provided that the dependence \( \Omega_{TX} = \Omega_{TX}(x^1) \) is specified. We shall restrict our attention to the case when in a small vicinity of the toroidal surface one can use the linear expansion

\[
\Omega_{TX} = \omega^2 \left( 1 - \frac{x^1 - x^1_{TX}}{\ell_{TX}} \right) \tag{21}
\]

where \( x^1_{TX} \) is a coordinate of the toroidal surface (on which \( \Omega_{TX} = \omega \)), \( \ell_{TX} \) is the inhomogeneity scale, and it is assumed
Leonovich and Mazur (1993) introduced the notations

\[ \varepsilon_{t,x} = \frac{r_{t,x}}{k_{t,x}^2}, \quad \varepsilon_{s,x} = 2r_{s,x} \varepsilon_{t,x}. \]

Remember that \( \varepsilon_{t,x} \) is a typical transverse wavelength of the large-scale mode near the toroidal surface. For the oscillations of interest with \( m \gg 1 \), it is rather small: \( \varepsilon_{t,x} \ll \varepsilon_{l,x} \). Let us introduce the dimensionless coordinate \( \xi = (x^l - x_{t,x}^l) / \varepsilon_{t,x} \), and a complex variable \( z = \xi + i \varepsilon_{s,x} \). Equation (25) then assumes the form

\[ x^l \partial^2 \partial \partial_{x^l} - \Gamma_{s,x} + \Gamma_{l,x} - \Gamma_{c,x} = 0. \] (25)

Here it is designated that \( x^l = \Lambda_{l,x}/\sqrt{\varepsilon_{t,x}} \), and the derivatives are taken in the variable \( z \). The dimensionless parameter \( x^l \) is small: \( |x^l| \ll 1 \). Taking into consideration that it generally is a complex one, we put \( x^l = |x^l| e^{i \theta} \). Since \( \text{Im} \Lambda_{l,x} \leq 0 \), it may be assumed that \( 0 \leq \theta \leq \pi \). To positive \( \Lambda_{l,x} \), there corresponds the value of \( \theta = 0 \), while the value of \( \theta = \pi \) corresponds to negative \( \Lambda_{l,x} \).

A solution of equation (25) is readily obtained using the Laplace method. A full set of linearly independent solutions is given by the integrals

\[ F_t(z) = \frac{1}{\pi} \int_{C_t} dt \exp \left( \frac{z^2}{3} t^3 + \frac{1}{3} t + z t \right). \] (26)

Each of the possible paths of integrations \( C_t \) in the plane of a complex variable \( t \) must be such that the function

\[ Z(t) = \frac{1}{i} \exp \left( \frac{z^2}{3} t^3 - \frac{1}{3} t \right) \]

takes, at its ends, equal values (or, for a closed path, it returns to the original value when the path is traced around). It is easy to see that the function \( Z(t) \to 0 \) when \( |t| \to \infty \). From these considerations it follows that the solutions of equation (25) are the integrals (26), provided that one of the contours \( C_1, C_2, \ldots, C_n \) are chosen as the path of integration, as shown in Fig. 2. Since there exist only four linearly independent solutions, these solutions involve three relationships that are readily established from the pattern

\[ F_1 = F_2 - F_3, \quad F_4 = F_2 - F_3, \quad F_5 = F_2 - F_3 + F_4. \]
A general solution is the linear superposition of any four linearly independent functions $F_k$.

The particular form of this superposition is determined by boundary conditions which must be satisfied by the solution in an asymptotically distant region, formally when $z \to \pm \infty$. For the solution of interest, these conditions can be formulated as follows. Firstly, this must be a bounded solution. This means that growing asymptotic representations must be absent in the opacity regions of both the large-scale and small-scale waves. Secondly, in the transmission region of the small-scale mode its asymptotic representation must be a wave that carries the energy from the resonance region (i.e. its group velocity must be directed from the toroidal surface to infinity). From the physical point of view, this signifies that, on the one hand, the kinetic wave is generated in the neighbourhood of the resonance surface as a result of the transformation of the large-scale mode. On the other hand, there are no kinetic waves that bring the energy from infinity, that is, waves generated by some external sources. These conditions fix the desired solution up to an arbitrary factor which is determined by the amplitude of the incident large-scale wave.

It appears that the conditions formulated are satisfied by the solution $F_1(z)$. To verify this, we consider the asymptotic representations of the function $F_1(z)$. Omitting standard calculations based on the saddle-point method (see Budden, 1961), we give the final result as

$$F_1(z) = z^{-1/4} \exp \left( -2i(-z)^{1/2} - i\frac{\pi}{4} \right)$$

$$+ \left( -\frac{z}{\mu} \right)^{-3/4} \exp \left[ -\frac{2}{3} \left( -\frac{z}{\mu} \right)^{3/2} + i\frac{\pi}{4} \right], \quad z \to -\infty$$

Here it is designated as $\mu = \frac{|z|^{2/3}}{|\lambda_\lambda|^{2/3}}, \mu_\lambda T$. From these asymptotic representations it is evident that the typical wavelength in the variable $z$ for the large-scale mode is unity, and for the small-scale mode it equals the value of $\mu$. In terms of the initial variable $\lambda$ they are, respectively, $\lambda T \propto \mu_\lambda T = |\lambda_\lambda|^{2/3} |\lambda_\lambda|$. Incidentally, the last assertion holds only for the sufficient smallness of the damping. For the large-scale wave this smallness implies $\varepsilon T \ll 1$. For the small-scale wave the condition is more rigorous

$$\delta_\lambda = \frac{\varepsilon T \lambda}{\mu} \approx \frac{2 \lambda_\lambda}{s_\lambda w_\lambda} \ll 1.$$  

If, however, the inverse inequalities $\varepsilon T \gg 1$ and $\delta_\lambda \gg 1$ are satisfied, then expressions (27) and (28) are inapplicable, and the typical scale of the oscillation is determined by dissipative parameters (see below).

From formulas (27) and (28) it follows that in the transmission region of the large-scale mode ($\zeta < 0$) this is a wave running toward the resonance surface, and the reflected wave is absent. The result of our previous paper is thereby reproduced, with the only difference being that the large-scale wave is now not absorbed on the toroidal surface but is transformed into the kinetic Alfvén wave.

The transformation effect is manifested most distinctly in the absence of the dissipation, that is, when $\gamma_\lambda = 0$ and at real values of $\lambda_\lambda$. For positive values of $\lambda_\lambda$, that is, when $\psi = 0$, from (27) and (28) we have

$$F_1(z) = \left( -\frac{z}{\mu} \right)^{-3/4} \exp \left[ -\frac{2}{3} \left( -\frac{z}{\mu} \right)^{3/2} + i\frac{\pi}{4} \right], \quad z \to -\infty$$

$$F_1(z) = z^{-1/4} \exp (-2z^{1/2})$$

$$+ \left( -\frac{z}{\mu} \right)^{-3/4} \exp \left[ -\frac{2}{3} \left( -\frac{z}{\mu} \right)^{3/2} - i\frac{\pi}{4} \right], \quad z \to \infty$$

and for $\Lambda_\lambda < 0$, that is, when $\psi = \pi$, we obtain

$$F_1(z) = \left( -\frac{z}{\mu} \right)^{-3/4} \exp \left[ -\frac{2}{3} \left( -\frac{z}{\mu} \right)^{3/2} + i\frac{\pi}{4} \right], \quad z \to -\infty$$

$$F_1(z) = z^{-1/4} \exp (-2z^{1/2})$$

$$+ \left( -\frac{z}{\mu} \right)^{-3/4} \exp \left[ -\frac{2}{3} \left( -\frac{z}{\mu} \right)^{3/2} - i\frac{\pi}{4} \right], \quad z \to \infty$$

$$F_1(z) = \left( -\frac{z}{\mu} \right)^{-3/4} \exp \left[ -\frac{2}{3} \left( -\frac{z}{\mu} \right)^{3/2} + i\frac{\pi}{4} \right], \quad z \to \infty.$$
Let us demonstrate that the transformation of the large-scale wave into a small-scale wave is a complete one, that is, the energy flux carried by the small-scale wave is equal to the energy flux brought by the large-scale wave. For this purpose, we use the expression for transverse components of the Poynting flux vector \( \mathbf{S}_i (i = 1, 2) \) obtained by Leonovich and Mazur (1993). A little manipulation on the formulas obtained in that paper yields

\[
\mathbf{S}_i = \frac{c^2}{8\pi} k_i^2 |r_i|^2 |V_i|^2.
\]

Let us consider the case \( \Lambda_2^2 > 0 \). According to the formulas from the cited paper, in the transmission region of the large-scale mode

\[
k_i = \frac{1}{j_{\Lambda_2}} (-z)^{\frac{1}{2}}, \quad r_i = \frac{\omega_j}{j_{\Lambda_2}} (-z)^{\frac{3}{2}}
\]

and from (30) we have

\[
|V_i|^2 = (-z)^{-\frac{1}{2}}.
\]

Hence

\[
\mathbf{S}_i = \frac{c^2}{8\pi} \frac{\omega_j}{j_{\Lambda_2}}.
\]

In the transmission region of the small-scale mode

\[
k_i = \frac{1}{j_{\Lambda_2}} \frac{z}{i}, \quad r_i = i k_1 \Lambda_2^2, \quad |V_i|^2 = \left(\frac{z}{j}\right)^{-3/2}.
\]

Whence

\[
\mathbf{S}_i = \frac{c^2}{8\pi} \frac{\omega_j}{j_{\Lambda_2}}.
\]

Similar calculations can also be made for \( \Lambda_2^2 < 0 \). In this case the transmission regions for the large- and small-scale waves coincide. Their phase velocities have the same sense of direction: toward the toroidal surface. However, when \( \Lambda_2^2 < 0 \) the group velocity of the small-scale wave is directed opposite to the phase velocity, it carries the energy away from the resonance surface. Qualitatively the structure of the mode on coordinate \( x' \) is presented in Fig. 3.

If the weak dissipation on the ionosphere is taken into account, then the amplitudes of the running waves decrease in the course of their propagation. By confining ourselves to the case \( \Lambda_2^2 > 0 \) and retaining only the leading asymptotic representations, from (30) we obtain

\[
F_1(z) = \left[ \left( \frac{x_{2T}}{x_{2T} - 1} \right)^{1/2} \exp \left( -2i \frac{x_{2T} - x^{1/2}}{x_{2T}} \right) \right.
\]

\[
-2i x_{2T} \left( \frac{1}{x_{2T} - x} \right)^{1/2} \left( \frac{\pi}{4} \right) - x_{2T} - x \gg x_{2T}, \quad (32a)
\]

\[
\left( \frac{x_{2T}}{x_{1T} - x} \right)^{1/2} \exp \left( \frac{2i}{3} \left( \frac{x^{1/2} - x_{1T}}{s_{1T}} \right)^{3/2} \right)
\]

\[
-2i \left( \frac{x^{1/2}}{s_{1T}} \right)^{1/2} \left( \frac{\pi}{4} \right) - x^{1/2} - x_{1T} \gg s_{1T}, \quad (32b)
\]

The full spatial structure of the wave field is determined by the potential

\[
\Phi = CF_1 T_\Lambda e^{i\psi} =
\]

\[
C \left( \frac{x^{1/2}}{j_x} \right)^{1/2} F_1 \left( \frac{x^{1/2} - x_{1T}}{j_x} \right) + i e^{\pi} \left( x^{1/2} - x \right) e^{i\phi}.
\]

Here we have passed (as in the paper of Leonovich and Mazur (1993)) from the function \( T_\Lambda \) to the function \( \phi \), where \( \phi = \left( pt + A \right)^{1/2} T_\Lambda \) with the advantage that it is dimensionless and, on the order of magnitude, unity. The constant \( C \) is determined by the amplitude of the incident large-scale wave. By comparing expression (33) with the corresponding expression from the cited paper in the region \( x_{1T} \ll x^{1/2}, -x^{1/2} \ll \Delta x_{1T} \), we obtain
Here, as before, the “cap” denotes the physical value of a corresponding quantity (say, $f_{,\phi} = \hat{g}_{,\phi}$), and the subscript “zero” refers to its equatorial value. For an explanation of the other symbols, the reader is referred to the cited paper.

From (33) and (34) we readily obtain the expressions for physical components of the wave’s electric field:

$$F_1 = -\frac{1}{\sqrt{g_1}} \hat{E}_{,\phi} = -\frac{\hat{g}_{,\phi}}{\hat{g}_{,\phi}} \hat{E}_{,\phi} - \frac{\sigma_0 A}{\sigma_{A0}} F_1 \left(\frac{\hat{g}_{,\phi}}{\hat{g}_{,\phi}} + i\omega_{A0}\right) F_1$$

$$F_2 = -\frac{i k_2}{\sqrt{g_2}} \phi = -i k_2 \hat{g}_{,\phi} \hat{E}_{,\phi} e^{i\omega \tau} - \frac{\sigma_0 A}{\sigma_{A0}} F_2 \left(\frac{\hat{g}_{,\phi}}{\hat{g}_{,\phi}} + i\omega_{A0}\right) F_2$$

The ratio of these components, on the order of magnitude, is

$$\frac{F_1}{F_2} \sim \frac{1}{k_2 \hat{g}_{,\phi}} \frac{\omega_{A0}}{\sigma_0 A} F_1.$$ 

As far as the ratio $F_1/F_2$ is concerned, its representative value near the toroidal surface, precisely where the kinetic dispersion is essential only, we have obtained an ordinary differential equation that describes the wave field structure in the direction normal to the magnetic shells (equation (11)).

(3) By solving this equation, we have obtained formulas that fully define the spatial structure of a standing monochromatic Alfven wave near the toroidal surface (formulas (33)-(35)).

From these formulas it follows that a relatively large-scale Alfven wave, generated by external sources in the neighbourhood of the poloidal surface and transferred, as a consequence of the curvilinear dispersion, toward the toroidal surface, undergoes in its vicinity a linear transformation into a small-scale kinetic Alfven wave. The transformation is a complete one: the reflected large-scale wave is absent, and the energy flux that is brought by the large-scale wave to the toroidal surface, is equal to the energy flux carried by the small-scale wave away from this surface. As a consequence of the Ohmic dissipation on the ionospheric terminations the kinetic Alfven wave attenuates slowly in the process of its propagation across the magnetic shells. The attenuation length is much larger than the transverse wavelength but is much smaller than the distance between the poloidal and toroidal surfaces. One is led to conclude that Alfven waves, after having transformed into the kinetic mode, are dissipated in the immediate vicinity of the toroidal surface.

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4. Conclusions

Let us formulate the results obtained in this study.

(1) We have obtained a partial differential equation that defines the spatial structure of a monochromatic axially small-scale ($m \gg 1$) Alfven wave in the axi-symmetric magnetosphere (equation (8)). It is a generalization to the analogous equation from our previous paper which includes kinetic dispersion effects of Alfven waves.

(2) Based on this partial differential equation, near the toroidal surface, precisely where the kinetic dispersion is essential only, we have obtained an ordinary differential equation that describes the wave field structure in the direction normal to the magnetic shells (equation (11)).

References


Leonovich, A. S. and Mazur, V. A., Resonance excitation of
