Pergamon

Planet. Space Sci., Vol. 43, No. 7, pp. 885–893, 1995

0032–0633/95 $9.50+0.00

0032–0633(94)00207–X

Linear transformation of the standing Alfven wave in an axisymmetric magnetosphere

A. S. Leonovich and V. A. Mazur

Institute of Solar-Terrestrial Physics, Irkutsk 33, P.O. Box 4026, 664033, Russia

Received 15 March 1994; revised 3 October 1994; accepted 3 October 1994

Abstract. In an attempt to further develop the theory of transversally small-scale standing Alfven waves, constructed by Leonovich and Mazur (*Planet. Space Sci.* **41**, 697–717, 1993) their spatial structures are investigated in the neighbourhood of the toroidal resonance surface. Near this surface, of important significance are the effects of finite Larmor radius of ions and of electron inertia which lead to the so-called kinetic dispersion of Alfven waves. It is shown that near the toroidal surface there occurs a total linear transformation of one kind of standing Alfven waves, whose dispersion is due to the curvature of geomagnetic field lines (these were thoroughly investigated in the above-cited paper), into standing Alfven waves of another type, kinetic waves. Formulas have been obtained which thoroughly define the spatial structure of the waves under consideration.

1. Introduction

This paper is a direct continuation of our earlier publication (Leonovich and Mazur, 1993) devoted to the theory of transversally small-scale standing Alfven waves in the axisymmetric magnetosphere. It should here be emphasized that in that paper the small-scale character of the waves was assumed not only in the direction normal to the magnetic shells (which is quite natural, by virtue of the inhomogeneity of plasma in this direction) but also in the azimuthal direction. In other words, waves with large azimuthal wave numbers, $m \gg 1$, were considered.

The theory was developed within the framework of ideal magnetic hydrodynamics. In this approximation in a homogeneous plasma and in a homogeneous magnetic field, Alfven waves are known to be devoid of transverse dispersion. But Leonovich and Mazur (1990) showed that in a magnetic field with curved field lines, transversally small-scale Alfven waves involve a specific transverse dispersion which leads to a slow (as compared with the Alfven velocity) displacement of standing Alfven waves across the magnetic shells. Consequences of this effect were studied in detail by Leonovich and Mazur (1993). They showed that a standing Alfven wave is generated by external sources near a given magnetic shell where its polarization has a poloidal character (the electric field oscillates azimuthally, and the magnetic field oscillates in the direction normal to the magnetic shell), displaces then to another shell where it has toroidal character (the magnetic field oscillates azimuthally, and the electric field oscillates in the direction normal to the magnetic shell), and is totally absorbed on this shell. We termed the first and second shells, respectively, "poloidal and toroidal resonance surfaces".

In our developed theory we used the WKB approximation in a coordinate normal to the magnetic shells. It turns out that the corresponding quasi-classical wave vector is zero on the poloidal surface and becomes infinite on the toroidal surface. Note that this fully agrees with the polarization of the wave. The extreme comminution of the transverse spatial structure of the field wave, as the wave approaches the toroidal surface, leads to two important consequences. On the one hand, there is a decrease of the contribution of the dispersion associated with the curvature of the field lines, which is manifested in that the transverse group velocity, caused by this dispersion, tends rapidly to zero. On the other hand, there is an increase of the role of the better known dispersion of the Alfven waves, caused by effects which are beyond the scope of an ideal MHD, namely the inertia of the electrons and the finite Larmor radius of the ions. Alfven waves, for which such a dispersion is important, received the name of kinetic waves (see, for example, Hasegawa and Uberoi (1982)). The combination of these two factors leads to the need to take into account the kinetic dispersion when investigating the spatial structure of a standing Alfven wave near the toroidal surface. This is the subject of the present study.

Correspondence to: A. S. Leonovich

Kinetic Alfven waves have already been addressed in numerous publications on hydrodynamical oscillations of the magnetosphere (Hasegawa, 1976; see also reviews by Southwood and Hughes, 1983; Goertz, 1984). However, those publications used a magnetospheric model in the form of a plane plasma sheet (a homogeneous magnetic field and a one-dimensionally inhomogeneous (in the transverse direction) plasma). Kinetic Alfven waves in the magnetospheric model with curved geomagnetic field lines and an inhomogeneous (in both the transverse and longitudinal directions) plasma were considered by Leonovich and Mazur (1989). They investigated the resonant excitation of a standing Alfven wave by monochromatic magnetosound penetrating into the magnetosphere from outside. Their treatment involved examining oscillations with small values of azimuthal wave number, $m \sim 1$, because only for such m magnetosound can penetrate deep into the magnetosphere. It was shown that magnetosound excites a kinetic Alfven wave in the neighbourhood of the toroidal surface, and from this surface the wave moves slowly away and is damped gradually due to the Ohmic dissipation at ionospheric terminations. This result differs substantially from the pattern of the event when $m \gg 1$. In this last case, as will be shown later, the Alfven wave that is generated far from the toroidal surface, undergoes—in its immediate vicinity—a linear transformation to the kinetic Alfven wave.

2. The derivation of the equation for the spatial structure of the Alfven wave

In this paper we will use the notions and notations introduced by Leonovich and Mazur (1993). In particular, the axisymmetrical magnetosphere will be described in terms of an orthogonal curvilinear coordinate system x^1, x^2, x^3, in which the coordinate x^1 characterizes the magnetic shell, the coordinate x^2 represents the field line on this shell (the azimuthal angle φ can be used as x^2), and the coordinate x^3 varies along the field line. The diagonal components of the metric tensor will be denoted by g_1, g_2, g_3, and $g = g_1 g_2 g_3$ is its determinant.

The disturbed electric field of a monochromatic wave obeys the equation

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = \frac{\omega^2}{c^2}\hat{\varepsilon}\mathbf{E} \qquad (1)$$

where $\hat{\varepsilon}$ is the dielectric constant tensor. The hydromagnetic oscillations of interest are low-frequency ones, $\omega \ll \omega_i$, where $\omega_i = eB/m_i c$ is the gyrofrequency of the ions, and are relatively large-scale ones, $\hat{k}_\perp \rho_i \ll 1$, where $\hat{k}_\perp$ is the physical meaning of the transverse wave vector, ρ_i being the Larmor radius of the ions. For such oscillations, the physical components (i.e. the components in a local Euclidean basis) of the dielectric constant tensor have the form (Akhiezer *et al.*, 1974)

$$\hat{\varepsilon}_{11} = \frac{c^2}{A^2}(1 - \tfrac{3}{4}\hat{k}_1^2\rho_i^2),$$

$$\hat{\varepsilon}_{22} = \frac{c^2}{A^2}(1 - \tfrac{3}{4}\hat{k}_2^2\rho_i^2),$$

$$\hat{\varepsilon}_{33} = \frac{c^2}{\omega^2\Lambda_s^2},$$

$$\hat{\varepsilon}_{12} = \hat{\varepsilon}_{21} = -\tfrac{3}{4}\hat{k}_1\hat{k}_2\rho_i^2,$$

$$\hat{\varepsilon}_{13} = \hat{\varepsilon}_{31} = \hat{\varepsilon}_{23} = \hat{\varepsilon}_{32} = 0. \qquad (2)$$

Here $A = B_0/\sqrt{(4\pi\rho)}$ is the Alfven velocity, $\hat{k}_1$ and $\hat{k}_2$ are the physical components of the wave vector

$$\Lambda_s^2 = \frac{\rho_s^2}{w(\omega/k_\parallel v_e)}, \quad \rho_s = \frac{v_s}{\omega_i}, \quad \rho_i = \frac{v_i}{\omega_i},$$

$$v_s = \left(\frac{T_e}{m_i}\right)^{1/2}, \quad v_e = \left(\frac{T_e}{m_e}\right)^{1/2}, \quad v_i = \left(\frac{T_i}{m_i}\right)^{1/2}$$

and $w(z)$ is a function well known in plasma physics (see Fried and Conte, 1961)

$$w(z) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{t\exp(-t^2/2)}{t-z}\,dt$$

$$= 1 - z\,e^{-z^2/2}\int_0^z e^{t^2/2}\,dt + i\left(\frac{\pi}{2}\right)^{1/2} z\,e^{-z^2/2}.$$

For real z, this function has the following limiting expressions:

$$w(z) = \begin{cases} 1 + i(\pi/2)^{1/2}z, & |z| \ll 1 \\ -\dfrac{1}{z^2} + i\left(\dfrac{\pi}{2}\right)^{1/2} z\,e^{-z^2/2}, & |z| \gg 1 \end{cases}. \qquad (3)$$

In the approximation of an ideal MHD $\rho_i = \Lambda_s = 0$. In this case relationships (1) and (2) describe, in a homogeneous plasma, the independent Alfven and magnetosound waves with the dispersion laws $\omega^2 = k_\parallel^2 A^2$ and $(k_\parallel^2 + k_\perp^2)A^2$, respectively. A specific property of the Alfven waves is the absence of the transverse dispersion: the frequency ω does not depend on $k_\perp$. When going beyond the scope of an ideal MHD, they are imparted a weak transverse dispersion. In a homogeneous plasma, from (1) and (2) it follows that

$$\omega^2 = k^2A^2(1 + k_\perp^2\Lambda^2), \quad \Lambda^2 = \Lambda_s^2 + \tfrac{3}{4}\rho_i^2. \qquad (4)$$

Such Alfven waves received the name kinetic waves. The corresponding dispersion will also be referred to as the kinetic dispersion here.

Even in the presence of a dispersion, the approximate equality $\omega \approx k_\parallel A$ is valid, and in the argument of the function w one may put

$$\frac{\omega}{k_\parallel v_e} = \frac{A}{v_e} \equiv \frac{s}{\rho_s} \equiv \frac{\beta_e}{(m_e/m_i)}$$

where $s = c/\omega_{pe}$ is the electron skin length, and $\beta_e = 8\pi n_0 T_e/B_0^2$ the ratio of electron to magnetic pressure. Hence

$$\Lambda_s^2 = \frac{\rho_s^2}{w(s/\rho_s)}.$$

In particular

$$\Lambda_s^2 = \begin{cases} -s^2, & s \gg \rho_s, \quad (\beta_e \ll m_e/m_i) \\ \rho_s^2, & s \ll \rho_s, \quad (\beta_e \gg m_e/m_i) \end{cases}.$$

When $s \sim \rho_s$ (i.e. $\beta_e \sim m_e/m_i$), the quantity Λ_s^2 is a complex one, and $\mathrm{Im}\,\Lambda_s^2 < 0$. From (4) it follows that this corresponds to the damping of the wave. It is caused by the Cherenkov resonance due to electrons which is effective by virtue of the relationship $v_e \sim A$. In the magnetospheric plasma the values of s and ρ_s are extremely small (varying from a few hundred metres to several tens of kilometres) compared with typical scales of the magnetosphere. Therefore, the kinetic dispersion has a role only for extremely small-scale waves in the transverse direction.

In an inhomogeneous plasma and an inhomogeneous magnetic field relationships (1) and (2) can be brought to a system of differential equations for covariant components of the electric field of the wave E_i ($i = 1, 2, 3$). These are related to the physical components $\hat{E}_i$ by the equality $\hat{E}_i = E_i/\sqrt{g_i}$. The quantities k_i in the expression for the dielectric constant tensor should be treated as the operators $\hat{k}_i = -\mathrm{i}(1/\sqrt{g_i})\nabla_i$, where $\nabla_i = \partial/\partial x^i$. Since the terms that contain the operators $\hat{k}_i$, play an important role for extremely small-scale waves, it may be assumed that they commute with functions that describe equilibrium parameters of the plasma and the magnetic field (i.e. the derivatives ∇ in these operators can be referred only to fields E_i).

From (1) and (2) when $k_\perp^2|\rho_s^2| \ll 1$ it is easy to obtain the expression for the longitudinal component E_3:

$$E_3 = \Lambda_s^2\left(\frac{1}{g_1}\nabla_1\nabla_3 E_1 + \frac{1}{g_2}\nabla_2\nabla_3 E_2\right). \tag{5}$$

After that, the system of equations for the transverse components E_1 and E_2 can be reduced to the form

$$(\hat{P}^{ij} + \hat{L}^{ij})E_j = 0 \tag{6}$$

where the operator $\hat{P}_{ij}$ has the same form as in Leonovich and Mazur (1993):

$$\hat{P}^{ij} = \tilde{\nabla}^i \frac{g_3}{\sqrt{g}} \tilde{\nabla}^j, \quad \tilde{\nabla}^i = (\nabla_2, -\nabla_1)$$

and $\hat{L}_{ij}$ differs from that in the previous paper by the presence of dispersion additions:

$$\hat{L}^{11} = \nabla_3 \frac{g_2}{\sqrt{g}} \nabla_3 + \frac{\sqrt{g}}{g_1}\frac{\omega^2}{A^2} + \frac{3}{4}\frac{\sqrt{g}\rho_i^2}{g_1^2}\frac{\omega^2}{A^2}\nabla_1^2 - \nabla_3 \frac{g_2}{g_1}\frac{\Lambda_s^2}{\sqrt{g}}\nabla_1^2\nabla_3$$

$$\hat{L}^{22} = \nabla_3 \frac{g_1}{\sqrt{g}} \nabla_3 + \frac{\sqrt{g}}{g_2}\frac{\omega^2}{A^2} + \frac{3}{4}\frac{\sqrt{g}\rho_i^2}{g_2^2}\frac{\omega^2}{A^2}\nabla_2^2 - \nabla_3 \frac{g_1}{g_2}\frac{\Lambda_s^2}{\sqrt{g}}\nabla_2^2\nabla_3$$

$$\hat{L}^{12} = \hat{L}^{21} = \frac{3}{4}\frac{\sqrt{g}\rho_i^2}{g_1 g_2}\nabla_1\nabla_2 - \nabla_3\frac{\Lambda_s^2}{\sqrt{g}}\nabla_1\nabla_2\nabla_3.$$

Using, as in the paper just cited above, the perturbation theory based on the transverse small-scale character of the oscillations, we find that in the main order of this theory the solution of system (6) has the form

$$E_i = -\nabla_i\Phi \tag{7}$$

where the potential Φ satisfies the equation

$$(\nabla_1 \hat{L}_1 \nabla_1 + \nabla_2 \hat{L}_p \nabla_2)\Phi + \frac{3}{4}\sqrt{g_\perp}\rho_i^2\frac{\omega^2}{A^2}\Delta_\perp^2\Phi - \frac{\partial}{\partial l}\sqrt{g_\perp}\Lambda_s^2\Delta_\perp^2\frac{\partial\Phi}{\partial l} = 0. \tag{8}$$

Here we have passed from the variable x^3 to the physical length along the field line l, whose differentials are related by the relationship

$$\mathrm{d}l = \sqrt{g_3}\,\mathrm{d}x^3$$

as well as using the notations

$$\hat{L}_1 = \frac{\partial}{\partial l}p\frac{\partial}{\partial l} + p\frac{\omega^2}{A^2}, \quad \hat{L}_p = \frac{\partial}{\partial l}\frac{1}{p}\frac{\partial}{\partial l} + \frac{1}{p}\frac{\omega^2}{A^2},$$

$$p = \left(\frac{g_2}{g_1}\right)^{1/2}, \quad \Delta_\perp = \frac{1}{g_1}\nabla_1^2 + \frac{1}{g_2}\nabla_2^2, \quad g_\perp = g_1 g_2.$$

Equation (8) describes the spatial structure of an Alfven wave. It differs from an analogous equation reported in Leonovich and Mazur (1993) by the presence of the two last terms.

The boundary condition on the ionosphere for the potential Φ has the same form as in the cited paper:

$$\Phi|_{l_\pm} = \mp\mathrm{i}\frac{c^2\cos\chi_\pm}{4\pi\Sigma_p^{(\pm)}}\left.\frac{\partial\Phi}{\partial l}\right|_{l_\pm} \tag{9}$$

Here $l_\pm$ represents the coordinates of the ionospheric ends of the field line, $\chi_\pm$ refers to angles they make with the local vertical, and $\Sigma_p^{(\pm)}$ corresponds to integral Pedersen conductivities of the conjugate ionospheres. Dispersion effects in (9) can be neglected because the dispersion parameters ρ_i and Λ_s on the ionosphere are much smaller than those in the magnetosphere.

In order to pass from the partial differential equation (8) to an ordinary differential equation that describes the mode structure near the toroidal surface, we avail ourselves of the perturbation theory based on the closeness of the desired solution to the toroidal mode. This means that this solution can be represented as

$$\Phi = [V_N(x^1)T_N(x^1, l) + \varphi_N]\,\mathrm{e}^{\mathrm{i}k_2x^2}. \tag{10}$$

Here k_2 is the azimuthal wave vector (if $x^2 = \varphi$ is the azimuthal angle, then $k_2 = m$ is the azimuthal wave number), T_N a toroidal wave function, and φ_N a small correction. The function T_N is the eigen-solution of the longitudinal problem

$$\hat{L}(\Omega_{TN})T_N \equiv \frac{\partial}{\partial l}p\frac{\partial T_N}{\partial l} + p\frac{\Omega_{TN}^2}{A^2}T_N = 0, \quad T_N|_{l_\pm} = 0.$$

Here $\Omega_{TN} = \Omega_{TN}(x^1)$ are toroidal eigenfrequencies. The desired function in (10) is the function $V_N(x^1)$ that defines the transverse structure of the mode. The equation describing it is the solvability condition for the correction φ_N.

Proceeding along similar lines as in Leonovich and Mazur (1993), we obtain

$$\omega^2\Lambda_N^2\frac{d^4V_N}{dx^{1^4}}+\frac{d}{dx^1}[(\omega+i\gamma_N)^2-\Omega_{TN}^2]\frac{dV_N}{dx^1} -k_2^2w_{TN}V_N=0. \quad (11)$$

Here

$$\Lambda_N^2=\oint\left[\frac{1}{g_1}(\Lambda_s^2+\tfrac{3}{4}\rho_i^2)+\frac{A^2}{2\omega^2}\frac{1}{p}\frac{\partial}{\partial\ell}p\frac{\partial}{\partial\ell}\frac{\Lambda_s^2}{g_1}\right]\frac{p}{A^2}T_N^2\,d\ell$$

$$w_{TN}=-\oint\left(\frac{\partial^2}{\partial\ell^2}\frac{1}{p}\right)T_N^2\,d\ell$$

and γ_N is the decrement of damping of the mode on the ionosphere, and the expression for it is given in the cited paper (it is assumed that $\gamma_N\ll\omega$). In that paper it is also emphasized that the value of w_{TN} is nonzero as a consequence of the curvature of geomagnetic field lines, and it is shown that for realistic magnetospheric models it is positive. The expressions for ρ_N^2 and w_{TN} simplify considerably for harmonics with $N\gg1$, when the WKB approximation in coordinate ℓ is applicable. In this case

$$\Lambda_N^2=\frac{1}{t_A}\oint\frac{\Lambda^2}{g_1}\frac{d\ell}{A}\equiv\left\langle\frac{\Lambda^2}{g_1}\right\rangle;\quad t_A=\oint\frac{d\ell}{A}, \quad (12)$$

$$w_{TN}=-\frac{1}{t_A}\oint\left(\frac{\partial^2}{\partial\ell^2}\frac{1}{p}\right)\frac{A}{p}d\ell.$$

Equation (11) differs from an analogous equation from our previous work by the presence of the first term. An important role is played by the sign of the dispersion parameter Λ_N^2. In the inner part of the magnetosphere where $\beta_e\ll m_e/m_i$ and, consequently, $s^2\gg\rho_i^2,\rho_s^2$ and $\Lambda_s^2\approx -s^2$ it is negative. In the outer magnetosphere where $\beta_e\gg m_e/m_i$ and $\Lambda_s^2\approx\rho_s^2$ it is positive. In the intermediate region this parameter is a complex one, and $\mathrm{Im}\,\Lambda_s^2<0$.

3. Linear transformation of a standing Alfven wave near the toroidal resonance surface

If the WKB approximation is applied to equation (11), that is, the solution is sought in the form

$$V_N\sim\exp(i\textstyle\int k_1\,dx^1) \quad (13)$$

then for the quasi-classical wave vector k_1 we obtain (by neglecting the damping on the ionosphere) the equation

$$\omega^2\Lambda_N^2k_1^4-(\omega^2-\Omega_{TN}^2)k_1^2-k_2^2w_{TN}=0. \quad (14)$$

Alternatively, this equation may be treated as a relationship that relates the local frequency ω to the wave vector k_1. By solving it for ω and taking into consideration that $|k_1^2\rho_N^2|\ll1$, we obtain a local dispersion equation

$$\omega^2=\Omega_{TN}^2+k_1^2\Lambda_N^2\Omega_{TN}^2-\frac{k_2^2w_{TN}}{k_1^2}. \quad (15)$$

The last two terms on the right-hand side represent dispersion corrections. For relatively large-scale waves, such that

$$k_1^2\ll\frac{k_2}{\Lambda_N}\frac{w_{TN}^{1/2}}{k_1^2} \quad (16)$$

one can put

$$\omega^2=\Omega_{TN}^2-\frac{k_2^2w_{TN}}{k_1^2}. \quad (17)$$

These waves were investigated in our previous paper and were referred to as small-scale waves. This implied that their transverse wavelength is much smaller than typical magnetospheric scales (it should be noted that the limit $k_1^2\to0$ cannot be considered in (17): this would imply violating the applicability condition for the WKB approximation; the relevant criteria were considered in the cited paper). In the present study the waves that satisfy condition (16), will be referred to as large-scale ones by reserving the term of the small-scall wave for the inverse case

$$k_1^2\gg\frac{k_2}{\Lambda_N}\frac{w_{TN}^{1/2}}{\omega}.$$

In this case

$$\omega^2=\Omega_{TN}^2(1+k_1^2\Lambda_N^2). \quad (18)$$

This dispersion equation is quite similar to equation (4) for kinetic Alfven waves. Based on (18) one can determine the group velocity of a small-scale standing Alfven wave in coordinate x^1:

$$v_{gN}^1=\frac{\partial\omega}{\partial k_1}=\Omega_{TN}k_1\Lambda_N^2. \quad (19)$$

Note that when $\Lambda_N^2>0$ (i.e. in a rather hot plasma, $\beta_e\gg m_e/m_i$) the signs of group velocity v_{gN}^1 and phase velocity $\omega/k_\perp$ coincide, and when $\Lambda_N^2<0$ (in a cold plasma, $\beta_e\ll m_e/m_i$) they are opposite.

We now return to the usual (for the WKB approximation) treatment of relationship (14) as an equation for k_1. From it we have

$$k_1^2=\frac{\omega^2-\Omega_{TN}^2\pm\sqrt{((\omega^2-\Omega_{TN}^2)^2+4k_2^2\Lambda_N^2\omega^2W_{TN})}}{2\omega^2\Lambda_N^2}. \quad (20)$$

This equality defines the function $k_1^2=k_1^2(x^1)$, provided that the dependence $\Omega_{TN}=\Omega_{TN}(x^1)$ is specified. We shall restrict our attention to the case when in a small vicinity of the toroidal surface one can use the linear expansion

$$\Omega_{TN}^2=\omega^2\left(1-\frac{x^1-x_{TN}^1}{\ell_N}\right) \quad (21)$$

where x_{TN}^1 is a coordinate of the toroidal surface (on which $\Omega_{TN}=\omega$), ℓ_N is the inhomogeneity scale, and it is assumed

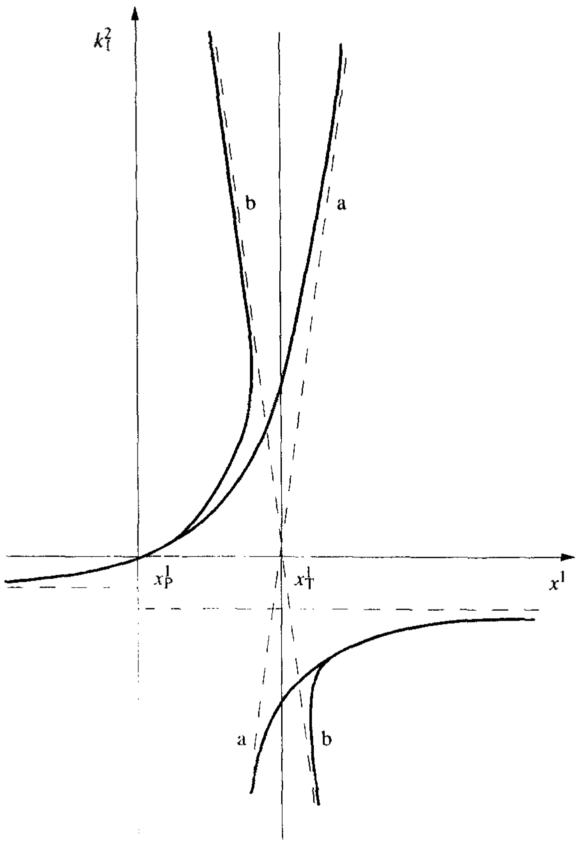

Fig. 1. The square of the quasi-classical wave vector k_1^2 plotted against the coordinate x^1. Dashed lines show the asymptotic representations $k_1^2 = (x^1 - x^1_{TN})/(l_N\Lambda_N^2)$. Case "a" corresponds to values of $\Lambda_N^2 > 0$, and case "b" corresponds to values of $\Lambda_N^2 < 0$

that $|x^1 - x^1_{TN}| \ll l_N$. On substituting (21) into (20), we obtain

$$k_1^2 = \frac{1}{2\Lambda_N^2}\left[\frac{x^1 - x^1_{TN}}{l_N} \pm \sqrt{\left(\left(\frac{x^1 - x^1_{TN}}{l_N}\right)^2 + 4k_2^2\Lambda_N^2\frac{w_{TN}}{\omega^2}\right)}\right]. \tag{22}$$

This dependence is plotted in Fig. 1.

Sufficiently far from the toroidal surface, when $|x^1 - x^1_{TN}| \gg k_2\Lambda_N(w_{TN}^{1/2}/\omega)/l_N$, the two roots in (22) assume the form

$$k_1^2 = -\frac{k_2^2 l_N w_{TN}}{\omega^2}\frac{1}{x^1 - x^1_{TN}}, \quad k_1^2 = \frac{x^1 - x^1_{TN}}{l_N\Lambda_N^2}. \tag{23}$$

The first of them describes a large-scale wave, whose dispersion is given by equation (17). The transmission region for it is located at $x^1 < x^1_{TN}$. The second root represents a small-scale (kinetic) wave with the dispersion law (18). The transmission region for it is located at $x^1 > x^1_{TN}$ if $\Lambda_N^2 > 0$, and at $x^1 < x^1_{TN}$ if $\Lambda_N^2 < 0$.

In the small vicinity of the toroidal surface, waves of one type transform to waves of another type. In order to investigate qualitatively this process and, in particular, to determine the transformation coefficient, it is necessary to have recourse to the original equation (1). On substituting expression (21) into it, we obtain

$$\Lambda_N^2\frac{d^4V_N}{dx^{1\,4}} + \frac{d}{dx^1}\left(\frac{x^1 - x^1_{TN}}{l_N} + 2i\frac{\gamma_N}{\omega}\right)\frac{dV_N}{dx^1} - \frac{k_2^2 w_{TN}}{\omega^2}V_N = 0. \tag{24}$$

Leonovich and Mazur (1993) introduced the notations

$$\lambda_{TN} = \frac{\omega^2}{k_2^2 l_N w_{TN}}, \quad \varepsilon_{TN} = 2\frac{l_N}{\lambda_{TN}}\frac{\gamma_N}{\omega}.$$

Remember that λ_{TN} is a typical transverse wavelength of the large-scale mode near the toroidal surface. For the oscillations of interest with $m \gg 1$, it is rather small: $\lambda_{TN} \ll l_N$. Let us introduce the dimensionless coordinate $\xi = (x^1 - x^1_{TN})/\lambda_{TN}$ and a complex variable $z = \xi + i\varepsilon_{TN}$. Equation (24) then assumes the form

$$\alpha^2 V_N^{(IV)} + zV_N'' + V_N' - V_N = 0. \tag{25}$$

Here it is designated that $\alpha^2 = \Lambda_N^2/l_N\lambda_{TN}^3$, and the derivatives are taken in the variable z. The dimensionless parameter α^2 is small: $|\alpha^2| \ll 1$. Taking into consideration that it generally is a complex one, we put $\alpha^2 = |\alpha^2|e^{-i\psi}$. Since Im $\Lambda_N^2 \leqslant 0$, it may be assumed that $0 \leqslant \psi \leqslant \pi$. To positive Λ_N^2, there corresponds the value of $\psi = 0$, while the value of $\psi = \pi$ corresponds to negative Λ_N^2.

A solution of equation (25) is readily obtained using the Laplace method. A full set of linearly independent solutions is given by the integrals

$$F_k(z) = \frac{1}{\sqrt{\pi}}\int_{C_k}\frac{dt}{t}\exp\left(\frac{\alpha^2}{3}t^3 + \frac{1}{t} + zt\right). \tag{26}$$

Each of the possible paths of integrations C_k in the plane of a complex variable t must be such that the function

$$Z(t) = \frac{1}{t}\exp\left(\frac{\alpha^2}{3}t^3 + \frac{1}{t}\right)$$

takes, at its ends, equal values (or, for a closed path, it returns to the original value when the path is traced around). It is easy to see that the function $Z(t) \to 0$ when $|t| \to \infty$ in the following sectors:

$$\frac{2\pi n + \psi}{3} + \frac{\pi}{6} < \arg t < \frac{2\pi n + \psi}{3} + \frac{\pi}{2}$$

where n is an arbitrary whole number. In Fig. 2 these sectors are shaded. Moreover, $Z(t) \to 0$ if $t \to 0$ proceeds so that Re $t < 0$ (say, along the negative semi-axis of the real t). From these considerations it follows that the solutions of equation (25) are the integrals (26), provided that one of the contours $C_1, C_2, \ldots, C_7$ are chosen as the path of integration, as shown in Fig. 2. Since there exist only four linearly independent solutions, these solutions involve three relationships that are readily established from the pattern

$$F_4 = F_2 - F_1, \quad F_5 = F_3 - F_2, \quad F_6 = F_3 - F_1 + F_2.$$

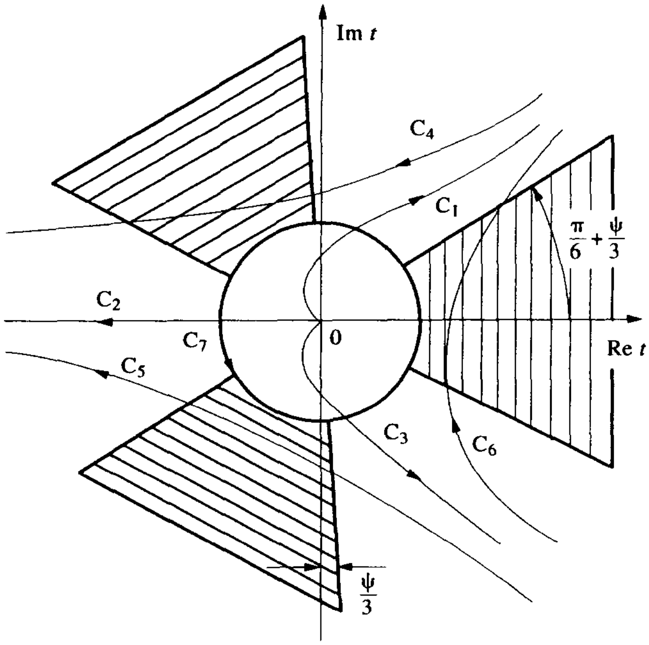

Fig. 2. The possible paths of integration in formula (26)

A general solution is the linear superposition of any four linearly independent functions F_k.

The particular form of this superposition is determined by boundary conditions which must be satisfied by the solution in an asymptotically distant region, formally when $z \to \pm\infty$. For the solution of interest, these conditions can be formulated as follows. Firstly, this must be a bounded solution. This means that growing asymptotic representations must be absent in the opacity regions of both the large-scale and small-scale waves. Secondly, in the transmission region of the small-scale mode its asymptotic representation must be a wave that carries the energy from the resonance region (i.e. its group velocity must be directed from the toroidal surface to infinity). From the physical point of view, this signifies that, on the one hand, the kinetic wave is generated in the neighbourhood of the resonance surface as a result of the transformation of the large-scale mode. On the other hand, there are no kinetic waves that bring the energy from infinity, that is, waves generated by some external sources. These conditions fix the desired solution up to an arbitrary factor which is determined by the amplitude of the incident large-scale wave.

It appears that the conditions formulated are satisfied by the solution $F_1(z)$. To verify this, we consider the asymptotic representations of the function $F_1(z)$. Omitting standard calculations based on the saddle-point method (see Budden, 1961), we give the final result as

$$F_1(z) = (-z)^{-1/4}\exp\left[-2i(-z)^{1/2} - i\frac{\pi}{4}\right] + \left(-\frac{z}{\mu}\right)^{-3/4}\exp\left[-\tfrac{2}{3}\left(\cos\frac{\psi}{2} + i\sin\frac{\psi}{2}\right)\left(-\frac{z}{\mu}\right)^{3/2} + i\frac{\pi}{2} - i\frac{\psi}{4}\right], \quad z \to -\infty \tag{27}$$

$$F_1(z) = z^{-1/4}\exp(-2z^{1/2}) + \left(\frac{z}{\mu}\right)^{-3/4} \times \exp\left[-\frac{2}{3}\left(\sin\frac{\psi}{2} - i\cos\frac{\psi}{2}\right)\left(\frac{z}{\mu}\right)^{3/2} - i\frac{\pi}{4} + i\frac{\psi}{4}\right], \quad z \to \infty. \tag{28}$$

Here it is designated as $\mu = |\alpha|^{2/3} = |\Lambda_N|^{2/3}\ell_N^{1/3}/\lambda_{TN}$. From these asymptotic representations it is evident that the typical wavelength in the variable z for the large-scale mode is unity, and for the small-scale mode it equals the value of μ. In terms of the initial variable x^1 they are, respectively, λ_{TN} and $s_N \equiv \mu\lambda_{TN} = |\Lambda_N|^{2/3}\ell_N^{1/3}$. Incidentally, the last assertion holds only for the sufficient smallness of the damping. For the large-scale wave this smallness implies $\varepsilon_{TN} \ll 1$. For the small-scale wave the condition is more rigorous

$$\delta_N \equiv \frac{\varepsilon_{TN}}{\mu} \cong 2\frac{\ell_N}{s_N}\frac{\gamma_N}{\omega} \ll 1. \tag{29}$$

If, however, the inverse inequalities $\varepsilon_{TN} \gg 1$ and $\delta_N \gg 1$ are satisfied, then expressions (27) and (28) are inapplicable, and the typical scale of the oscillation is determined by dissipative parameters (see below).

From formulas (27) and (28) it follows that in the transmission region of the large-scale mode ($z < 0$) this is a wave running toward the resonance surface, and the reflected wave is absent. The result of our previous paper is thereby reproduced, with the only difference being that the large-scale wave is now not absorbed on the toroidal surface but is transformed into the kinetic Alfven wave.

The transformation effect is manifested most distinctly in the absence of the dissipation, that is, when $\gamma_N = 0$ and at real values of Λ_N^2. For positive values of Λ_N^2, that is, when $\psi = 0$, from (27) and (28) we have

$$F_1(z) = (-z)^{-1/4}\exp\left[-2i(-z)^{1/2} - i\frac{\pi}{4}\right] + \left(-\frac{z}{\mu}\right)^{-3/4}\exp\left[-\frac{2}{3}\left(-\frac{z}{\mu}\right)^{3/2} + i\frac{\pi}{2}\right], \quad z \to -\infty$$

$$F_1(z) = z^{-1/4}\exp(-2z^{1/2}) + \left(\frac{z}{\mu}\right)^{-3/4}\exp\left[-\frac{2}{3}i\left(\frac{z}{\mu}\right)^{3/2} - i\frac{\pi}{4}\right], \quad z \to \infty \tag{30}$$

and for $\Lambda_N^2 < 0$, that is, when $\psi = \pi$, we obtain

$$F_1(z) = (-z)^{-1/4}\exp\left[-2i(-z)^{1/2} - i\frac{\pi}{4}\right] + \left(-\frac{z}{\mu}\right)^{-3/4}\exp\left[-\frac{2}{3}i\left(-\frac{z}{\mu}\right)^{3/2} + i\frac{\pi}{4}\right], \quad z \to -\infty$$

$$F_1(z) = z^{-1/4}\exp(-2z^{1/2}) + \left(\frac{z}{\mu}\right)^{-3/4}\exp\left[-\frac{2}{3}\left(\frac{z}{\mu}\right)^{3/2}\right], \quad z \to \infty. \tag{31}$$

Let us demonstrate that the transformation of the large-scale wave into a small-scale wave is a complete one, that is, the energy flux carried by the small-scale wave is equal to the energy flux brought by the large-scale wave. For this purpose, we use the expression for transverse components of the Poynting flux vector $\bar{S}^i (i = 1, 2)$ obtained by Leonovich and Mazur (1993). A little manipulation on the formulas obtained in that paper yields

$$\bar{S}^1 = \frac{c^2}{8\pi} k_1^2 v_N^1 |V_N|^2.$$

Let us consider the case $\Lambda_N^2 > 0$. According to the formulas from the cited paper, in the transmission region of the large-scale mode

$$k_1^2 = \frac{1}{\lambda_{TN}^2}(-z)^{-1}, \quad v_N^1 = \frac{\omega\lambda_{TN}^2}{l_N}(-z)^{3/2}$$

and from (30) we have

$$|V_N|^2 = (-z)^{-1/2}.$$

Hence

$$\bar{S}^1 = \frac{c^2}{8\pi}\frac{\omega}{l_N}.$$

In the transmission region of the small-scale mode

$$k_1^2 = \frac{1}{\lambda_{TN}^2}\frac{z}{\mu}, \quad v_N^1 = \omega k_1 \Lambda_N^2, \quad |V_N|^2 = \left(\frac{z}{\mu}\right)^{-3/2}.$$

Whence

$$\bar{S}^1 = \frac{c^2}{8\pi}\frac{\omega}{l_N}.$$

Similar calculations can also be made for $\Lambda_N^2 < 0$. In this case the transmission regions for the large- and small-scale waves coincide. Their phase velocities have the same sense of direction: toward the toroidal surface. However, when $\Lambda_N^2 < 0$ the group velocity of the small-scale wave is directed opposite to the phase velocity, it carries the energy away from the resonance surface. Qualitatively the structure of the mode on coordinate x^1 is presented in Fig. 3.

If the weak dissipation on the ionosphere is taken into account, then the amplitudes of the running waves decrease in the course of their propagation. By confining ourselves to the case $\Lambda_N^2 > 0$ and retaining only the leading asymptotic representations, from (30) we obtain

$$F_1(z) =$$

$$\begin{cases} \left(\dfrac{\lambda_{TN}}{x_{TN}^1 - x^1}\right)^{1/4} \exp\left[-2i\left(\dfrac{x_{TN}^1 - x^1}{\lambda_{TN}}\right)^{1/2} - \varepsilon_{TN}\left(\dfrac{\lambda_{TN}}{x_{TN}^1 - x^1}\right)^{1/2} - i\dfrac{\pi}{4}\right], & x_{TN}^1 - x^1 \gg \lambda_{TN} \quad (32a) \\ \left(\dfrac{s_N}{x^1 - x_{TN}^1}\right)^{1/4} \exp\left[\dfrac{2}{3}i\left(\dfrac{x^1 - x_{TN}^1}{s_N}\right)^{3/2} - \delta_N\left(\dfrac{x^1 - x_{TN}^1}{s_N}\right)^{1/2} - i\dfrac{\pi}{4}\right], & x^1 - x_{TN}^1 \gg s_N. \quad (32b) \end{cases}$$

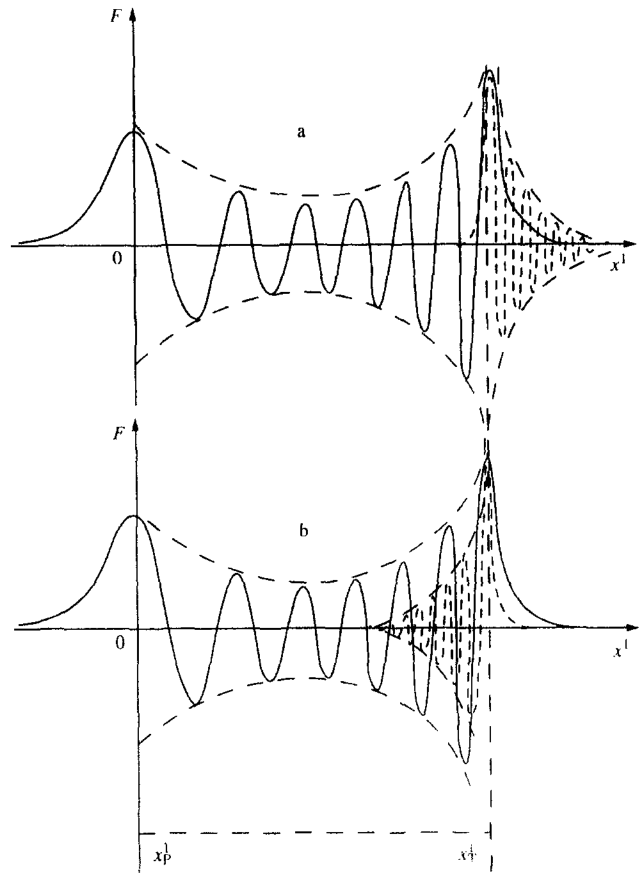

Fig. 3. Spatial structure of the potential Φ versus coordinate x^1. Solid line presents large-scale mode and shaded line small-scale mode and envelopes of amplitude. Case (a) corresponds to $\Lambda_N^2 > 0$ and case (b) to $\Lambda_N^2 < 0$

The typical attenuation length of the small-scale wave $s_N/\delta_N^2 \sim (\Lambda_N^2/l_N)(\omega^2/\gamma_N^2)$, albeit much larger than the wavelength s_N, is, however, much smaller (at realistic values of parameters) than the distance between the poloidal and toroidal surfaces. Thus, after having been transformed into the small-scale mode, the Alfven wave is dissipated in a small vicinity of the toroidal surface.

If the parameter Λ_N^2 is essentially complex such that the value of ψ is rather far from the values of $\psi = 0$ and π, then —as is apparent from (27) and (28)— the typical scale of attenuation of the small-scale mode coincides with its wavelength s_N.

The full spatial structure of the wave field is determined by the potential

$$\Phi = CF_1 T_N e^{ik_2x^2} =$$

$$C\left(\frac{A}{pt}\right)^{1/2} F_1\left(\frac{x^1 - x_{TN}^1}{\lambda_{TN}} + i\varepsilon_{TN}\right) r_N(x^1, l)\, e^{ik_2x^2}. \quad (33)$$

Here we have passed (as in the paper of Leonovich and Mazur (1993)) from the function T_N to the function $r_N = (pt_A/A)^{1/2} T_N$ with the advantage that it is dimensionless and, on the order of magnitude, unity. The constant C is determined by the amplitude of the incident large-scale wave. By comparing expression (33) with the corresponding expression from the cited paper in the region $\lambda_{TN} \ll x_{TN}^1 - x^1 \ll \Delta x_N^1$, we obtain

$$C = \sqrt{\pi}\hat{\lambda}^{(0)}_{PN}\tilde{E}\,e^{i\varphi - \Gamma}\left(\frac{p_0 t_A}{A_0}\right)^{1/2}. \tag{34}$$

Here, as before, the "cap" denotes the physical value of a corresponding quantity (say, $\hat{\lambda}_{PN} = \sqrt{g_1}\lambda_{PN}$), and the subscript "zero" refers to its equatorial value. For an explanation of the other symbols, the reader is referred to the cited paper.

From (33) and (34) we readily obtain the expressions for physical components of the wave's electric field:

$$\hat{E}_1 = -\frac{1}{\sqrt{g_1}}\frac{\partial\Phi}{\partial x^1} = -\sqrt{\pi}\frac{\hat{\lambda}^{(0)}_{PN}}{\hat{\lambda}^{(0)}_{TN}}\tilde{E}\,e^{i\tilde{\varphi}-\Gamma}\left(\frac{\sigma_0 A}{\sigma A_0}\right) \times F_1'\left(\frac{x^1 - x^1_{TN}}{\lambda_{TN}} + i\varepsilon_{TN}\right)r_N$$

$$\hat{E}_2 = -\frac{ik_2}{\sqrt{g_2}}\varphi = -i\sqrt{\pi}\hat{k}^{(0)}_2\hat{\lambda}^{(0)}_{PN}\tilde{E}\,e^{i\tilde{\varphi}-\Gamma}\frac{p_0}{p}\left(\frac{\sigma_0 A}{\sigma A_0}\right) \times F_1\left(\frac{x^1 - x^1_{TN}}{\lambda_{TN}} + i\varepsilon_{TN}\right)r_N. \tag{35}$$

The ratio of these components, on the order of magnitude, is

$$\frac{\hat{E}_1}{\hat{E}_2} \sim \frac{1}{\hat{k}^{(0)}_2\hat{\lambda}^{(0)}_{TN}}\frac{F_1'}{F_1}.$$

As far as the ratio F_1'/F_1 is concerned, its representative value near the toroidal surface depends on the relative role of the effects of the small-scale dispersion and dissipation. If the former effect has a more important role, then—as follows from (31)—$F'/F \sim 1/\mu$. If the dissipation is more important, then $F'/F \sim 1/\varepsilon_{TN}$ (see our previous paper). By comparing these quantities, we arrive at the conclusion that if

$$\left(\frac{\Lambda_N}{\ell_N}\right)^{2/3} \gg \frac{\gamma_N}{\omega} \tag{36}$$

then the dispersion effect is more important; otherwise, the dissipation effect in the ionosphere predominates. Under conditions of the Earth's magnetosphere either case can occur on different magnetic shells (for more details, see Leonovich and Mazur (1989)). If condition (36) is satisfied, then

$$\frac{\hat{E}_1}{\hat{E}_2} \sim \frac{1}{\hat{k}^{(0)}_2\hat{s}^{(0)}_N}$$

with the inverse inequality

$$\frac{\hat{E}_1}{\hat{E}_2} \sim \frac{1}{\hat{k}^{(0)}_2\hat{\ell}^{(0)}_N}\frac{\omega}{\gamma_N}$$

being satisfied. In either case the oscillation is a toroidal one: $\hat{E}_1 \gg \hat{E}_2$.

4. Conclusions

Let us formulate the results obtained in this study.

(1) We have obtained a partial differential equation that defines the spatial structure of a monochromatic azimuthally small-scale ($m \gg 1$) Alfven wave in the axisymmetric magnetosphere (equation (8)). It is a generalization to the analogous equation from our previous paper which includes kinetic dispersion effects of Alfven waves.

(2) Based on this partial differential equation, near the toroidal surface, precisely where the kinetic dispersion is essential only, we have obtained an ordinary differential equation that describes the wave field structure in the direction normal to the magnetic shells (equation (11)).

(3) By solving this equation, we have obtained formulas that fully define the spatial structure of a standing monochromatic Alfven wave near the toroidal surface (formulas (33)–(35)).

(4) From these formulas it follows that a relatively large-scale Alfven wave, generated by external sources in the neighbourhood of the poloidal surface and transferred, as a consequence of the curvilinear dispersion, toward the toroidal surface, undergoes in its vicinity a linear transformation into a small-scale kinetic Alfven wave. The transformation is a complete one: the reflected large-scale wave is absent, and the energy flux that is brought by the large-scale wave to the toroidal surface, is equal to the energy flux carried by the small-scale wave away from this surface. As a consequence of the Ohmic dissipation on the ionospheric terminations the kinetic Alfven wave attenuates slowly in the process of its propagation across the magnetic shells. The attenuation length is much larger than the transverse wavelength but is much smaller than the distance between the poloidal and toroidal surfaces. One is led to conclude that Alfven waves, after having transformed into the kinetic mode, are dissipated in the immediate vicinity of the toroidal surface.

Acknowledgement. We are grateful to Mr V. G. Mikhalkovsky for his assistance in preparing the English version of the manuscript. This work was partially supported by a grant from the Russian Fundamental Investigation Foundation (RFFI 94-05-16126-a).

References

Akhiezer, A. I., Akhiezer, I. A., Polovin, R. V., Sitenko, A. G. and Stepanov, K. N., *Plasma Electrodynamics* (edited by A. I. Akhiezer), p. 212. Nauka, Moscow, 1974.

Budden, K. G., *Radio Waves in the Ionosphere*. p. 300. Cambridge University Press, London, 1961.

Fried, B. D. and Conte, S. D., *The Plasma Dispersion Functions*. Academic Press, New York, 1961.

Goertz, C. K., Kinetic Alfven waves on auroral field lines. *Planet. Space Sci.* **32,** 1381, 1984.

Hasegawa, A., Particle acceleration by MHD surface wave and formation aurora. *J. geophys. Res.* **81,** 5083, 1976.

Hasegawa, A. and Uberoi, C., The Alfven waves. U.S. Department of Energy, 1982.

Leonovich, A. S. and Mazur, V. A., Resonance excitation of standing Alfven waves in an axisymmetric magnetosphere (monochromatic oscillations). *Planet. Space Sci.* **37**, 1095, 1989.

Leonovich, A. S. and Mazur, V. A., The spatial structure of poloidal Alfvén oscillations of an axisymmetric magnetosphere. *Planet. Space Sci.* **38**, 1231, 1990.

Leonovich, A. S. and Mazur, V. A., A theory of transverse small-scale standing Alfvén waves in an axially symmetric magnetosphere. *Planet. Space Sci.* **41**, 697–717, 1993.

Southwood, D. J. and Hughes, W. J., Theory of hydromagnetic waves in the magnetosphere. *Space Sci. Rev.* **35**, 301, 1983.