JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 100, NO. A6, PAGES 9527–9534, JUNE 1, 1995

On the propagation of transversally small-scale standing Alfven waves in a three-dimensionally inhomogeneous magnetosphere

Dmitrii Yu. Klimushkin, Anatolii S. Leonovich, and Vitalii A. Mazur

Institute of Solar-Terrestrial Physics, Irkutsk, Russia

Abstract. In terms of a three-dimensionally inhomogeneous model of the magnetosphere, we have investigated theoretically the process of propagation across the geomagnetic field of monochromatic standing Alfven waves that are transversally small-scale ones. For this purpose we introduced an orthogonal coordinate system tied in a natural manner to the geomagnetic field. It is shown that a three-dimensionally inhomogeneous magnetosphere is composed of sectors that differ from each other by the propagation character of the waves with a different sign of azimuthal wave number. The study revealed the existence of transversally small-scale waves produced near the toroidal surface in one sector and absorbed near this same surface in another sector of the magnetosphere. These phenomena have no analog in the theory of the axisymmetric magnetosphere where oscillations come into being near the poloidal surface and propagate toward the toroidal surface.

1. Introduction

It is impossible to construct a full theory for magnetohydrodynamic (MHD) oscillations of the Earth's magnetosphere, without allowing consistently for the three-dimensional inhomogeneity of the near-terrestrial plasma. A one-dimensionally inhomogeneous model is known to make it possible to take into account the plasma inhomogeneity in the radial direction [*Southwood*, 1974; *Chen and Hasegawa*, 1974]. Two-dimensionally inhomogeneous models (for example, an axisymmetric model) were studied in reasonably greater detail; they permit the inclusion of the plasma inhomogeneity in the radial and longitudinal (along the geomagnetic field) directions as well as the curvature of magnetic field lines [*Southwood and Kivelson*, 1986; *Walker*, 1987; *Taylor and Walker*, 1987; *Leonovich and Mazur*, 1989, 1993; *Chen and Cowley*, 1989]. A variety of interesting results were also obtained by investigating three-dimensionally inhomogeneous models; thus *Inhester* [1986] and *Schultze-Berge et al.* [1992] showed that the resonance excitation of standing Alfven wave by magnetosound occurs also in the three-dimensionally inhomogeneous case. Yet the study of MHD wave propagation in a three-dimensionally inhomogeneous magnetosphere has just been initiated, and the present paper is an attempt to make a step forward in this direction.

It should be noted that most of the aforementioned publications considered waves with values of azimuthal wave number $m \sim 1$, with their typical propagation feature being Alfven resonance. Waves with $m \gg 1$ have received relatively little attention. Such preference seems to be caused by the fact that such waves are much more rarely recorded experimentally compared to waves $m \sim 1$ [*Anderson et al.*, 1990; *Engebretson et al.*, 1988; *Takahashi and Anderson*, 1992; *Takahashi et al.*, 1990; *Walker et al.*, 1982; *Ostwald et al.*, 1993], as well as by the fact that the case of $m \gg 1$ is much more difficult to investigate theoretically. It is customary to assume that waves with large m are always recorded in the form of poloidal pulsations, in which the wave's magnetic field and the plasma oscillate largely in a radial direction. However, as was shown by *Leonovich and Mazur* [1993], poloidal waves, while propagating across magnetic shells, transform rather rapidly into toroidal ones. Consequently, part of the toroidal pulsations which are usually associated with $m \sim 1$ oscillations can actually have values of azimuthal wave number much larger than unity. Thus the contribution of the Alfven oscillations with $m \gg 1$ to an overall picture can be significantly larger than is generally appreciated.

This paper is an extension to the theory of monochromatic Alfen waves with large (but not infinite) values of azimithal wave number m in an axisymmetric magnetosphere, as suggested by *Leonovich and Mazur* [1993] (hereinafter referred to as paper 1). The main corollaries of the theory are the following. There exist two distinct L shells: a poloidal resonance surface on which the frequency of the wave is equal to the frequency of purely poloidal oscillations (i.e., oscillations of the shell in the radial direction), and a toroidal resonance surface on which the frequency of the wave is equal to the frequency of purely toroidal oscillations (at which the

Paper number 94JA03233.
0148-0227/95/94JA-03233$05.00

disturbed magnetic field and the plasma oscillate in the azimuthal direction). The Alfven wave is excited near the poloidal surface and propagates farther toward the toroidal surface. The mode is totally absorbed near the toroidal shell. Hence a transparency region of the waves concerned, outside which they do not propagate, lies between the resonance surfaces.

In this paper we consider how the conclusions of paper 1 change in a three-dimensionally inhomogeneous model of the magnetosphere. We investigate the behavior of the constant-phase lines (characteristics) of the transversally small-scale Alfven oscillations which are also the flux lines of the wave's energy. The paper is organized as follows. In section 2 we introduce a system of coordinates suitable for investigating a three-dimensionally inhomogeneous magnetosphere. Section 3 presents the basic equations defining the wave's structure which are studied qualitatively in section 4. The main results of this work are reported in section 5. A number of geometrical notions used in section 2 are discussed in the appendix.

2. The Coordinate System

Before developing the equations, we must introduce the coordinates to be used in this paper. A natural requirement which we impose on the coordinate system, is that the system must be as convenient as possible for a description of the Alfven waves in a three- dimensionally inhomogeneous magnetosphere. Such is the coordinate system that satisfies the following two conditions: (1) the geomagnetic field, **B**, lines are coordinate lines, and (2) the coordinate system is an orthogonal one; i.e., the coordinate surfaces $x^1 = c_1, x^2 = c_2$, and $x^3 = c_3$ produce a triorthogonal system. (Three families of curved surfaces are said to produce a triorthogonal system if any two surfaces of different families intersect at right angles.)

The advantages of such a choice are first that (in the approximation of ideal magnetohydrodynamics) the electric field of a hydromagnetic wave turns out to have only two components (see the beginning of the next section) and, second, that nondiagonal components of the metric tensor disappear.

Consider a magnetosphere model which permits us to introduce a coordinate system satisfying these conditions. The construction of such a coodinate system will be based on the Dupin theorem known from differential geometry [*Rashevsky*, 1950]. According to this theorem, lines along which surfaces of some one property of a triorthogonal system intersect with surfaces of two other families are lines of maximum and minimum curvature of these surfaces (see appendix). Hence it follows that coordinate lines of the coordinate system under consideration are lines of curvature of coordinate surfaces. We choose surfaces everywhere orthogonal to field lines as coordinate surfaces $x^3 =$const (Figure 1). Lines of maximum curvature of the surfaces $x^3 =$ const will be the

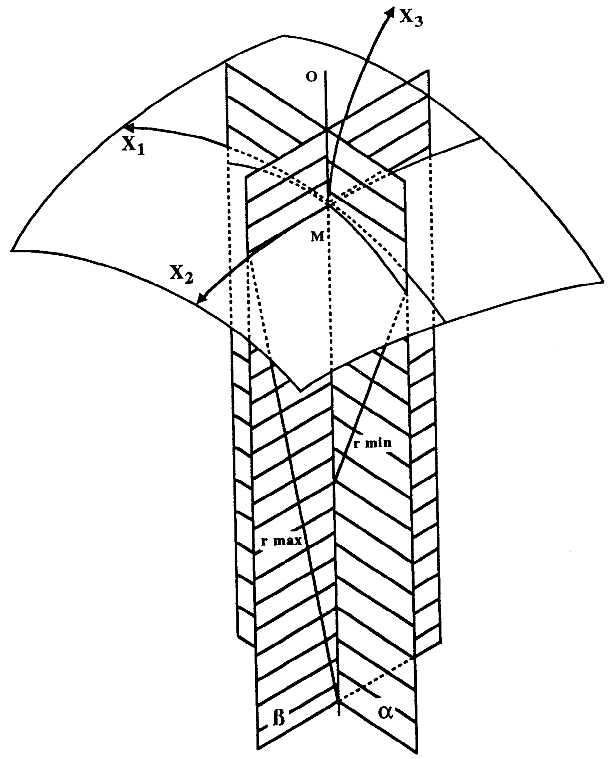

Figure 1. Scheme for constructing an orthogonal coordinate system. The coordinate lines x^3 coincide with the field lines. The figure shows a certain surface $x^3 =$ const; M, an arbitrary point above it; and O, normal to this surface at point M. The planes α and β contain the normal OM. They are chosen such that the lines of their intersection with the surface have a maximum r and minimum r radii of curvature at point M. The directions of these lines at each point specify the directions of the lines of minimum and maximum curvature of the surfaces, which, generally speaking, do not coincide with the lines of intersection but are only tangent to them. The lines of maximum and minimum curvature are the coordinate lines x^1 and x^2.

coordinate lines x^1, and lines of minimum curvature will be the coordinate lines x^2. (The coordinate line x^i is the line on which two other coordinates, except x^i, are constant.) It follows from the Dupin theorem that field lines are the lines of curvature of the surfaces $x^1 =$ const and $x^2 =$ const because they are the lines of intersection of these surfaces. The resulting system of coordinate surfaces satisfies the conditions described in the beginning of the section. Owing to the unique definition of the lines of curvature, our constructed coordinate system is determined uniquely as accurately as the transformations $x^1 = f(x^1)$, $x^2 = f(x^2)$, and $x^3 = f(x^3)$, which do not modify the form of the coordinate lines and the coordinate surfaces; this factor is important for the purposes of the subsequent discussion. The simultaneous specification of the coordinates x^1 and x^2 determines the field line, and the coordinate x^3 determines the point above it. The length element ds in this coordinate system is specified by a quadratic form

$$ds^2 = g_1 {dx^1}^2 + g_2 {dx^2}^2 + g_3 {dx^3}^2$$

where $g_i = g_{ii}(x^1, x^2, x^3)$ are the diagonal components of the metric tensor (the others are all zero by orthogonality). In the particular case of an axisymmetric magnetosphere, the coordinate x^1, introduced above, is a radial coordinate (for example, the McIlwain parameter L), and the coordinate x^2 is an azimuthal one (for example, the azimuthal angle ϕ).

Hamada [1962] constructed a coordinate system which satisfied condition 1 formulated earlier in this section, in which constant pressure surfaces are coordinate lines surfaces $x^1 = \text{const}$, and current lines and field lines are coordinate lines x^2 and x^3. Obviously, when $\mathbf{B} \cdot \mathbf{J} = 0$ this system also satisfies condition 2 (because $\nabla P = \mathbf{J} \times \mathbf{B}$). Since it follows from our preceding reasoning that surfaces $x^i = c_i$ are uniquely determined, we arrive at the conclusion that when $\mathbf{J} \neq 0$, lines of curvature of surfaces $x^3 = \text{const}$ should coincide with constant pressure lines and current lines.

3. The Basic Equations

We now turn directly to the wave theory. A disturbed electromagnetic field of a monochromatic oscillation of frequency ω is described by Maxwell's equations

$$\nabla \times \mathbf{E} = i\,\frac{\omega}{c}\,\mathbf{B}, \;\; \nabla \times \mathbf{B} = -i\,\frac{\omega}{c}\,\hat{\varepsilon}\,\mathbf{E}, \qquad (1)$$

where $\hat{\varepsilon}$ is the dielectric permittivity tensor. We will be treating the plasma in the approximation of ideal magnetohydrodynamics and neglecting the thermal motion of particles; the "physical" components of this tensor in our chosen coordinate system are

$$\hat{\varepsilon}_{11} = \hat{\varepsilon}_{22} = \frac{c^2}{A^2}, \;\; \hat{\varepsilon}_{33} = -\infty,$$

where $A = B_0/\sqrt{4\pi\rho}$ is the Alfven velocity. In view of infinitely large quantity of $\hat{\varepsilon}_{33}$ (which corresponds to an infinite plasma conductivity along the magnetic field), the component of a disturbed electric field E_3 of the wave is zero. As was shown in paper 1, the solution to (1) can be sought in the form

$$\mathbf{E}_\perp = -\nabla_\perp \Phi, \qquad (2)$$

where $\nabla_\perp$ is a two-dimensional transversal gradient, $\nabla_\perp = (\partial_1, \partial_2)$. The boundary condition on the ionosphere, under the assumption of its infinite conductivity, is written as

$$\Phi|_{x^3_\pm} = 0, \qquad (3)$$

where $x^3_\pm$ is the coordinate of intersection of the field line with the ionosphere of the magnetoconjugate hemispheres. On substituting (2) into (1), one may obtain the basic equation of our theory (see paper 1),

$$\left[\partial_1 \hat{L}_T(\omega)\partial_1 + \partial_2 \hat{L}_P(\omega)\partial_2\right]\Phi = 0, \qquad (4)$$

where

$$\hat{L}_T(\omega) = \frac{\partial}{\partial l} p \frac{\partial}{\partial l} + p\frac{\omega^2}{A^2}$$

$$\hat{L}_P(\omega) = \frac{\partial}{\partial l}\frac{1}{p}\frac{\partial}{\partial l} + \frac{1}{p}\frac{\omega^2}{A^2}$$

are the toroidal and poloidal operators, respectively. Here l is the physical length along the field line whose element $dl = \sqrt{g_3}dx^3$; we introduced the designation

$$p = \sqrt{\frac{g_2}{g_1}}.$$

It should be noted that (4) is an approximation assuming the small-scale length limit (as was done in paper 1).

In the theory under consideration, it is of fundamental importance that the value of p varies along the field line because otherwise the toroidal and poloidal operators would be simply proportional to each other. It is the dependence of p on l (a necessary condition for which is the curvature of the field lines) that determines the transverse propagation of the oscillations involved (for more detail, see *Leonovich and Mazur* [1990]). Because p and A are functions of coordinates, the operators $\hat{L}_P$ and $\hat{L}_T$ are also functions of x^1, x^2, and l. The eigenfunctions of these operators, referred to in paper 1, as the toroidal T_N and poloidal P_N eigenfunctions, respectively, satisfy the boundary conditions on the ionosphere similar to (3):

$$T_N, P_N\,|_{l_\pm} = 0,$$

where N is the number of half waves which are present along the field line.

Furthermore, let the eigenfrequencies of the toroidal and poloidal operators be designated, respectively, as $\Omega_{TN}(x^1, x^2)$ and $\Omega_{PN}(x^1, x^2)$. The surfaces that satisfy the implicit equations

$$\omega = \Omega_{TN}(x^1, x^2), \;\; \omega = \Omega_{PN}(x^1, x^2) \qquad (5)$$

will be called, as in paper 1, the toroidal and poloidal resonance surfaces. The distance between the resonance surfaces at the equator varies from a few hundred to several thousand kilometers. Note that when p is independent of the longitudinal coordinate, these surfaces would simply coincide.

We now turn to (4). Let the disturbance be represented as

$$\Phi = \exp iQ(x^1, x^2, l).$$

Because the relationships

$$\left|\frac{1}{\sqrt{g_1}}\frac{\partial\Phi}{\partial x^1}\right| \gg \left|\frac{\partial\Phi}{\partial l}\right|, \;\; \left|\frac{1}{\sqrt{g_2}}\frac{\partial\Phi}{\partial x^2}\right| \gg \left|\frac{\partial\Phi}{\partial l}\right|$$

are satisfied for transversally small-scale waves, to solve the problem of (3) and (4), it is possible to use the WKB approximation in the coordinates x^1, x^2. As shown in paper 1, in this approximation the asymptotic expansion of a quasi-classical phase Q has the form

$$Q(x^1, x^2, l) = Q_0(x^1, x^2) + Q_1(x^1, x^2, l),$$

where the main-order term Q_0 is a function only of transverse coordinates. By denoting $\exp(iQ_1) = H$, the disturbance is represented as

$$\Phi = H(x^1, x^2, l)\ \exp iQ_0(x^1, x^2). \tag{6}$$

The quasi-classical wave vector components are

$$k_1(x^1, x^2) = \frac{\partial Q_0}{\partial x^1},\ \ k_2(x^1, x^2) = \frac{\partial Q_0}{\partial x^2}. \tag{7}$$

We introduce the quantity

$$\kappa(x^1, x^2) = k_1/k_2.$$

On substituting (6) into (4) and (3), it is easy to obtain the equation

$$\left[\kappa^2 \hat{L}_T(\omega) + \hat{L}_P(\omega)\right] H = 0 \tag{8}$$

and the respective boundary condition

$$H\ |_{l_\pm} = 0. \tag{9}$$

With the specified x^1, x^2 and ω, these relationships can be treated as an eigenvalue problem for the quantity κ^2. Accordingly, the dependence of the eigenfunction $H = H_N(x^1, x^2, l, \omega)$ on l describes the longitudinal structure of the standing Alfven wave. Further, let the eigenvalues of κ^2 be $\kappa_N^2(x^1, x^2, \omega)$, i.e., $\kappa = \pm\kappa_N$ (the quantity κ_N is taken to be positive). Then a first-order partial differential equation for the phase Q_0 follows from (7):

$$\frac{\partial Q_0}{\partial x^1} \mp \kappa_N \frac{\partial Q_0}{\partial x^2} = 0. \tag{10}$$

The characteristics of this equation, i.e., the lines in the manifold $\{x^1, x^2\}$ along which $Q_0 = \text{const}$, are defined by an ordinary differential equation [*Farlow*, 1982]

$$\frac{dx^2}{dx^1} = \pm\kappa_N(x^1, x^2), \tag{11}$$

to give immediately the relationship

$$k_1 dx^1 + k_2 dx^2 = 0, \tag{12}$$

where the differentials dx^1 and dx^2 are taken along the characteristics.

It is easy to ascertain the physical meaning of the constant phase lines. To do this, we calculate the components of the transverse group velocity vectors of the oscillations concerned. We introduce the function $\omega_N = \omega_N(x^1, x^2, \kappa)$ inverse to the function $\kappa_N = \kappa_N(x^1, x^2, \omega)$. By differentiating this function, we obtain by definition

$$v_N^1 = \frac{\partial}{\partial k_1}\omega_N(x^1, x^2, k_1/k_2) = \frac{1}{k_2}\frac{\partial \omega_N}{\partial \kappa}$$

$$v_N^2 = \frac{\partial}{\partial k_2}\omega_N(x^1, x^2, k_1/k_2) = -\frac{k_1}{k_2^2}\frac{\partial \omega_N}{\partial \kappa}.$$

It follows from these relationships that

$$v_N^1\, k_1 + v_N^2\, k_2 = 0, \tag{13}$$

i.e., the characteristic is a line along which the transverse group velocity vector of the transversally small-scale Alfven waves is directed. In paper 1 it is shown that the transverse contravariant components of the Poynting vector, integrated over the volume of a flux tube with unit dimensions in the coordinates x^1 and x^2, are proportional to v^1 and v^2. This means that the characteristics are rays in the manifold $\{x^1, x^2\}$ along which the hydromagnetic wave energy flows. The plus and minus signs in (10) and (11) correspond to the waves running in the opposite directions along x^2 if their direction along x^1 is specified. It will be recalled once again that the wave is a standing one in the geomagnetic field direction.

4. A Qualitative Investigation of the Characteristic Equation

In order to solve the characteristic equation (11), it is necessary to solve the problem of (8) and (9), but owing to the complicated dependence of equilibrium parameters on coordinates, this is a tedious problem, better suited to numerical solution. However, to gain a qualitative understanding of the behavior of the characteristics, there is no need to do so. It will suffice to make the following remark. The problem of (8) and (9) can also be treated as a problem for eigenvalues of the quantity ω given κ. Consider two particular cases: $\kappa \to 0$ and $\kappa \to \infty$. In the former case, (8) becomes

$$\hat{L}_P(\omega)H = 0.$$

This equality (allowing for the boundary condition (9)) is satisfied when $\omega = \Omega_{PN}(x^1, x^2)$. Since the physical formulation of the problem is such that a wave with a given frequency is considered, we arrive at the conclusion that κ is zero on the poloidal surface. And conversely, when $\kappa \to \infty$, (8) reduces to

$$\hat{L}_T(\omega)H = 0.$$

This equality is satisfied on the toroidal surface.

It follows from (11) that near the poloidal surface, characteristics run along the coordinate surface $x^1 = \text{const}$, while near the toroidal surface they run along the surface $x^2 = \text{const}$. (From here on, while on the subject of coordinate and resonance surfaces, we will mean their projection onto a two-dimensional manifold of transverse coordinates.) It should be emphasized once again that the position of coordinate surfaces is uniquely defined by the magnetic field geometry. It turns out that a knowledge of the behavior of characteristics near resonance surfaces is sufficient to establish the main behavioral features of the characteristics throughout the entire transparency region.

In the case of an axisymmetric magnetosphere the toroidal and poloidal frequencies depend only on the coordinate x^1, the resonance surfaces coincide with the coordinate lines $x^1 = \text{const}$. This leads to a rather simple picture of the energy fluxes, analyzed in detail in paper 1: the wave energy generated near the poloidal surface flows out at a right angle, flows farther toward the toroidal surface, and changes its direction in such a way that the wave flows into the toroidal surface along a line tangent to it. Propagation of waves with $m > 0$ toward increasing x^2 is no different from propagation of waves with $m < 0$ toward decreasing x^2; corresponding lines of constant phase (characteristics) are the same in shape. The wave field pattern is independent of the azimuthal coordinate (Figure 2a).

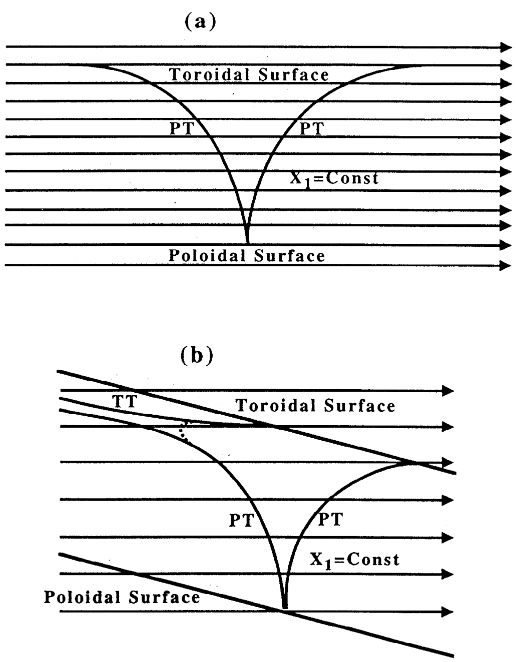

Figure 2. A schematic representation of the characteristics. Coordinate lines and resonance surfaces are shown by straight lines. (a) The axisymmetric magnetosphere. All characteristics correspond to type PT; i.e., they commence on the poloidal surface and terminate on the toroidal surface. (b) A sector of the three-dimensionally inhomogeneous magnetosphere in which $\mu < 0$. PT characteristics correspond to the sign $\kappa < 0$. Commencing in this magnetospheric sector, they can terminate either in the same sector or in another sector but with the same sign of μ. In the case $\kappa > 0$ both type PT and type TT characteristics are present, which commence and terminate on the toroidal surface. It is easy to see in the figure that the depicted parts of the characteristic with $\kappa > 0$ cannot be connected (as shown by dots), without coming into conflict with the condition $\kappa > 0$; i.e., they are necessarily different characteristics. TT characteristics terminate on the toroidal surface in a sector with different sign of μ.

If, however, there is no axial symmetry, then the pattern becomes much more complicated. In this case the poloidal and toroidal frequencies become dependent not only on x^1 but also on x^2; therefore resonance surfaces do not coincide with surfaces $x^1 = \text{const}$. The magnetosphere turns out to be composed of sectors which differ from each other by the sign of the angle of slope of the toroidal resonance surface to surface $x^1 = \text{const}$; i.e., the angle μ. In Figure 2b, for example, this angle is negative; if the toroidal surface is tilted in the opposite direction, it is positive. Obviously, such sectors are even in number.

The importance of this difference between the sectors becomes clear if it is recalled that near the toroidal surface the characteristic is tangent to line $x^1 = \text{const}$. Consider the propagation of a wave with $\kappa < 0$ in a magnetospheric sector with $m < 0$ (Figure 2b). As follows from (12), the characteristic corresponding to this wave passes between resonance surfaces from left to right because $dx^2 > 0$ holds for such a characteristic when $dx^1 > 0$. As is evident from Figure 2b, this wave changes smoothly the direction of its motion and enters the toroidal surface along line $x^1 = \text{const}$.

For waves with $\kappa > 0$, the propagation pattern differs drastically. A wave with $\kappa > 0$ must propagate from right to left because $dx^2 < 0$ occurs when $dx^1 > 0$. Even a cursory examination of Figure 2a shows that while remaining in a sector with $\mu < 0$, the wave with $\kappa > 0$ cannot reach the toroidal surface if it does not change its sign (shown by dots in Figure 2b) , but a change of sign is possible only in two cases: (1) if at some point κ goes to zero, or (2) if the characteristic has a kink. However, as was pointed out in the beginning of this section, κ vanishes solely on the poloidal surface, whose position is uniquely defined by equation (5). Therefore a change in propagation direction of the wave, accompanied by a change of sign of κ, is impossible. The characteristic also cannot undergo a kink because to the breakpoint there must correspond a singularity of (11), which is in fact not involved in this equation: its right-hand side is a regular one.

Thus we are left with only one logically conceivable possibility: the wave with $\kappa > 0$ cannot reach the toroidal surface in a sector with $\mu < 0$. Consequently, this wave propagates farther, into a sector with an opposite sign of μ where it is indeed provided with an opportunity to enter the toroidal surface along coordinate surface $x^1 = \text{const}$, without changing the sign of κ in this case. On the other hand, there must also exist a characteristic with $\kappa > 0$ which enters the toroidal surface in a sector with $\mu < 0$ because there exists an appropriate solution of equation (11). For this characteristic, there also exists the only possibility: after leaving the toroidal surface along line $x^1 = \text{const}$, it may proceed farther, into a sector with $\mu > 0$, where it again reaches the toroidal surface while entering it tangentially to coordinate line $x^1 = \text{const}$. Characteristics that start on the poloidal (P) surface and terminate on the toroidal (T) surface will be assigned to the PT type in what follows. This is the only type of characteristic which is possible in the axisymmetric magnetosphere. Characteristics that start on the toroidal surface and terminate also on it (but now in a different magnetospheric sector) will be referred to as TT type; they have no analog in the axisymmetric case. Note that the PT and TT characteristics with the same sign cannot either intersect or be connected with each other by a smooth curve (Figure 2b, dots) for reasons pointed out above.

The same treatment can also be undertaken for a sector with a positive angle μ between the toroidal and the coordinate x^1 =const surfaces. Obviously, to waves with $\kappa > 0$ there correspond PT characteristics; i.e., their propagation proceeds in a qualitatively similar manner as in the case of the wave propagation in the axisymmetric magnetosphere, whereas to waves with $\kappa < 0$ there correspond PT characteristics, which start in a given sector but terminate in a sector with $\mu < 0$, and TT characteristics. As is evident, axial symmetry of the near-terrestrial environment entails asymmetry in the wave propagation in each of the magnetospheric sectors.

As an illustration of the aforesaid, we give Figure 3b, showing a system of characteristics for $\kappa > 0$, calculated for a simple magnetosphere model in which the coordinate lines are concentric circles; both resonance surfaces also are concentric circles (again, the case in point is the manifold $\{x^1, x^2\}$), but their common center M does not coincide with the origin of the coordinate O. The following model of κ was chosen:

$$\kappa_N = \left(\frac{r - r_P}{r_T - r}\right)^{1/2},$$

where r is a coordinate reckoned from the center of the resonance surfaces which takes values of r_{T} and r_{P} on the toroidal and poloidal surfaces, respectively. Such a behavior of κ near the resonance surfaces is consistent with (8), from which using the perturbation method it is possible to get

$$\kappa_N^2 \sim \left(\omega^2 - \Omega_{PN}^2\right)$$

near the poloidal surface, and

$$\kappa_N^2 \sim \left(\Omega_{TN}^2 - \omega^2\right)^{-1}$$

near the toroidal surface (see section 4 of paper 1). Figure 3b clearly shows a whimsical picture of the characteristics as described above; we call the reader's attention to a qualitative difference of this figure from Figure 3a, which shows the characteristics in an axisymmetric magnetosphere. In Figure 3b, examples of constant-phase lines in a three-dimensionally inhomogeneous magnetosphere are numbered in order of increasing coordinate x^2 (designated as ϕ in the figure). Characteristic 1 starts at $\phi = 0$; it lies wholly in a sector with $\mu > 0$ (sector I) and belongs to the PT type.

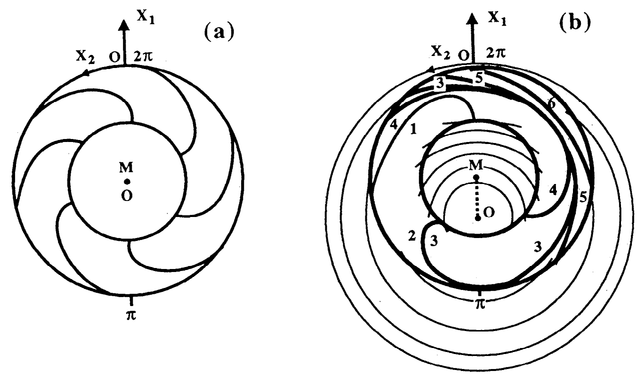

Figure 3. Characteristics (energy flux lines) at a fixed sign of κ in the cases of (a) the axisymmetric magnetospheric model and (b) the three-dimensionally-inhomogeneous model as described in the text. In Figure 3a, only PT characteristics occur, while in Figure 3b both PT and TT characteristics are present. Figure 3b is characterized by an asymmetry in the behavior of the characteristics in the sectors $0° - 180°$ and $180° - 360°$. The different types of characteristics, 1–6, are described in the text.

Characteristic 2 (PT type) also starts in sector I and terminates on the toroidal surface just on the boundary of sector I when $\phi = \pi$. All characteristics that start and terminate in sector I lie between characteristics 1 and 2 and are classified as PT type. Characteristic 3 (PT type) starts on the poloidal surface immediately following characteristic 2. It cannot terminate in sector II (for which $\mu < 0$) and passes throughout this sector, terminating in sector I. All subsequent characteristics that start on the poloidal surface between characteristics 3 and 1 terminate in sector I and are PT type; they are exemplified by characteristic 4, which starts on the poloidal surface in sector II and terminates on the toroidal surface in sector I. Characteristics 5 and 6 are classified as the TT type; they start on the toroidal surface in sector I and terminate on the same surface, but now in sector II. Characteristic 3 is a separatrix which separates sets of characteristics of the PT and TT types. Note that although in some portion of Figure 3b characteristics 4 and 5 merge visually with the separatrix, they lie in fact on opposite sides of it, while characteristic 6 passes away from the separatrix. Of course, in the real magnetosphere the number of sectors can be more than two, and they can be different in shape. However, the main behavioral features of the characteristics are illustrated wholly adequately by the example considered.

5. Conclusion

The results of the present work may be briefly summarized as follows.

1. We have suggested a coordinate system tailored to a description of the Alfven waves in the three-dimensionally inhomogeneous magnetosphere; and have demonstrated the uniqueness of the system of coordi-

nate surfaces under natural conditions imposed upon the coordinate system.

2. It has been shown that the three-dimensionally inhomogeneous magnetosphere is composed of sectors that differ from each other by the character of propagation of Alfven waves with a different sign of the azimuthal wave number. The propagation of the waves of the same sign κ as μ proceeds qualitatively similarly to the propagation of transversally small-scale waves in an axisymmetric magnetosphere: the wave is produced near the poloidal surface and travels toward the toroidal surface, where (in the same magnetospheric sector) it is absorbed. As far as the wave with opposite sign μ is concerned, it has to traverse a significant pathway before it is absorbed near the toroidal surface already in the sector with an opposite slope of the toroidal surface to the coordinate surface.

3. Unlike the axisymmetric model, the three-dimensionally inhomogeneous magnetosphere can allow for the existence of transversally small-scale waves produced (and absorbed) near the toroidal surface (in the axisymmetric magnetosphere near the toroidal surface, waves with small m could be produced, which are not azimuthally small-scale ones, while waves with large m could be produced only near the poloidal surface).

The spatial structure of the transversally small-scale waves in the three-dimensionally inhomogeneous magnetosphere will be considered elsewhere.

Appendix

In order to construct a coordinate system satisfying two conditions of section 2, remember the definition of the notion of lines of curvature and some of their properties [*Rashevsky,* 1950]. Consider a small area of an arbitrary curved surface near a certain point M (see Figure 1). Construct the normal O to this surface at point M and pass a section by a plane containing straight line O. The line of intersection of the plane and the surface will have a certain radius of curvature r. We vary the value of r by rotating the cutting plane about the axis O. In particular, along some directions, r will assume its minimum and maximum values; these directions are normal to each other. Lines along which the curvature radius of the surface is extreme are called the lines of curvature. With the exception of two particular cases (a sphere and a plane), lines of curvature of surfaces are determined uniquely.

Even if the surface is not entirely a spherical or plane one, it can include points at which a minimum and maximum radii of curvature are equal to each other, namely, points of rounding and flattening (in the latter case these radii are infinitely large). At these points, the directions of curvature are not defined. But these directions are uniquely determined at any points pertaining to an infinitesimal vicinity of the points of rounding and flattening. Therefore lines of curvature can also be drawn through the points of rounding and flattening by uniquely determining them in this way at these points as well.

To avoid misunderstanding, we must emphasize that in the general case, lines of curvature do not coincide with lines of intersection of the surface with a given plane, although they have one tangent line in common.

Acknowledgments. This work is partially supported by the Russian Foundation for Fundamental Research (grant 94-05-16126-a). We are grateful to V.G. Mikhalkovsky for his assistance in preparing the English version of the manuscript and for typing the text. We are grateful to S.M. Churilov for useful discussions of the coordinate system.

The Editor thanks R.E. Denton and A.D.M. Walker for their assistance in evaluating this paper.

References

Anderson, B.J., M.J. Engebretson, S.P. Rounds, L.J. Zanetti, and T.A. Potemra, A statistical study of Pc 3-5 pulsations observed by the AMPTE/CCE magnetic field experiment, 1, Occurrence distributions, *J. Geophys. Res.*, *95*, 10,495, 1990.

Chen, L., and S.C. Cowley, On field line resonances of hydromagnetic Alfven waves in dipole magnetic field, *Geophys. Res. Lett.*, *16*, 895, 1989.

Chen, L., and A. Hasegawa, A theory of long period magnetic pulsations, 1, *J. Geophys. Res.*, *79*, 1024, 1974.

Engebretson, M.J., L.J. Zanetti, T.A. Potemra, D.M. Klumpar, R.J. Strangeway, and M.N. Acuna, Observations of intense ULF pulsation activity near the geomagnetic equator during quiet times, *J. Geophys. Res.*, *93*, 12,795, 1988.

Farlow, S.J., *Partial Differential Equations for Scientists and Engineers*, John Wiley, New York, 1982.

Hamada, S., Hydromagnetic equilibria and their proper coordinates, *Nuc. Fusion*, *2*, 23, 1962.

Inhester, B., Resonance absorption of Alfven oscillations in a nonaxisymmetric magnetosphere, *J. Geophys. Res.*, *91*, 1509, 1986.

Leonovich, A.S., and V.A. Mazur, Resonance excitation of standing Alfven waves in an axisymmetric magnetosphere (monochromatic oscillations), *Planet. Space Sci.*, *37*, 1095, 1989.

Leonovich, A.S., and V.A. Mazur, The spatial structure of poloidal Alfven oscillations of an axisymmetric magnetosphere, *Planet. Space Sci.*, *38*, 1231, 1990.

Leonovich, A.S., and V.A. Mazur, A theory of transverse small-scale standing Alfven waves in an axially symmetric magnetosphere, *Planet. Space Sci.*, *41*, 697, 1993.

Ostwald, P.M., F.W. Menk, B.J. Fraser, C.L. Waters, and P.W. McNabb, Spatial and temporal characteristics of 15-100 mHz ULF-waves recorded across a low-latitude azimuthal array, *Ann. Geophys.*, *11*, 742, 1993.

Rashevsky, P. K., *The Course of a Differential Geometry* (in Russian), Gosudarstvennoye izdatelstvo tekhniko-teoreticheskoi literatury, Moscow, 1950.

Schultze-Berge, S., S. Cowley, and L. Chen, Theory of field line resonances of standing Alfven waves in three-dimensional inhomogeneous plasma, *J. Geophys. Res.*, *97*, 3219, 1992.

Southwood, D.J., Some features of field line resonances in the magnetosphere, *Planet. Space Sci.*, *22*, 483, 1974.

Southwood, D.J., and M.G. Kivelson, The effect parallel inhomogeneity of magnetospheric hydromagnetic wave coupling, *J. Geophys. Res.*, *91*, 6871, 1986.

Takahashi, K., and B.J. Anderson, Distribution of ULF-energy ($f < 80$ mHz) in inner magnetosphere: A statistical analysis of AMPTE CCE magnetic field data, *J. Geophys. Res.*, *97*, 10,751, 1992.

Takahashi, K., R.W. McEntire, A.T.Y. Lui, and T.A. Potemra, Ion flux oscillations with a radially polarized transversy Pc 5 magnetic pulsation, *J. Geophys. Res.*, *95*, 3717, 1990.

Taylor, J.P.H, and A.D.M. Walker, Theory of magnetospheric standing hydromagnetic waves with large azimuthal wave number, 2, Eigenmodes of the magnetosonic and Alfven oscillations., *J. Geophys. Res.*, *92*, 10,046, 1987.

Walker, A.D.M., Theory of magnetospheric standing hydromagnetic waves with large azimuthal wave number, 1, Coupled magnetosonic and Alfven waves, *J. Geophys. Res.*, *92*, 10,039, 1987.

Walker, A.D.M., R.A. Greenwald, A. Korth, and G. Kremser, STARE and GEOS 2 observations of a storm time Pc 5 ULF pulsation, *J. Geophys. Res.*, *87*, 9135, 1982.

D. Yu. Klimushkin, A. S. Leonovich, and V. A. Mazur, Institute of Solar-Terrestrial Physics, P.O. Box 33, Irkutsk 664 033, Russia. (e-mail: root@sitmis.irkutsk.su)

(Received June 16, 1994; revised December 2, 1994; accepted December 2, 1994.)