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RESONANCE EXCITATION OF STANDING ALFVÉN WAVES IN AN AXISYMMETRIC MAGNETOSPHERE (NONSTATIONARY OSCILLATIONS)

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Abstract—A theoretical study is made of the excitation—in an axisymmetric magnetosphere—of standing Alfvén waves by a nonstationary fast magnetosound which has a broad spectrum of frequencies. We have obtained formulae describing the space-and-time behaviour of Alfvén waves. An analysis is made of the role of a weak transverse dispersion of Alfvén waves and of their damping in the ionosphere. Particular emphasis is given to the important case of stochastic oscillations. Some examples are considered, which may be useful for modelling of Pi2, SSC and Pc3 phenomena.

1. INTRODUCTION

This paper is a direct continuation of a previous article by these authors (Leonovich and Mazur, 1989). Our intention here is to make a further attempt to develop the theory concerned, namely we examine the transition from monochromatic to nonstationary oscillations which have a wide spectrum. Such a transition appears to be important for oscillations of any type, but it is especially relevant to the Alfvén waves of our interest. A monochromatic source excites an Alfvén wave in the narrow vicinity of a resonance magnetic shell; hence, a broadband source whose spectrum involves a frequency range, must excite a corresponding range of resonance shells. This means that the space-and-time behaviour of monochromatic and broadband Alfvén waves will differ drastically.

Generation of Alfvén oscillations by a broadband source was considered in a number of earlier papers (Chen and Hasegawa, 1974; Krylov *et al.*, 1981; Hasegawa *et al.*, 1983; Allan *et al.*, 1986). However, they used a simple magnetosphere model in the form of a flat plasma layer in a homogeneous magnetic field. Our objective here is to investigate nonstationary Alfvén waves in an axisymmetric model of the magnetosphere. In addition, we take account of the transverse dispersion and dissipation of Alfvén waves; these effects are partly or completely ignored in the papers cited above.

According to resonance theory, the field of a magnetosound wave serves as a source for Alfvén waves. In this paper, as in the previous one, it is considered given. According to the character of the time behaviour it seems appropriate to divide all broadband magnetosound oscillations into two large classes. One should include nonstationary oscillations with a deterministic behaviour in time, for which the coordinate- and time-dependence of the field can, at least in principle, be considered known. Such oscillations are exemplified by magnetosound waves produced during SSC, magnetic substorms, and other nonstationary processes of a large scale. The other class involves stochastic oscillations generated by different instabilities such as the Kelvin–Helmholtz instability on the magnetopause (Kivelson and Pu, 1984) or an instability of a proton flux reflected from the front of a bow shock (Gul'elmi, 1984). The time-dependence of the field of such oscillations is a random function for which only certain statistic characteristics can be specified. Accordingly, statistic characteristics of an Alfvén wave are subject to a definition. This paper will consider both classes of oscillations.

The purpose of this paper is to derive general formulae which define the space-and-time evolution of nonstationary Alfvén oscillations. In order to apply these formulae for interpreting the particular kinds of geomagnetic pulsations, it is necessary to make special investigations; so we shall confine ourselves only to some examples having an illustrative character.

2. THE SPACE-AND-TIME BEHAVIOUR OF A NONSTATIONARY ALFVÉN WAVE

In the present paper we shall rely on results reported by Leonovich and Mazur (1989) and shall use the notations introduced therein. The transition from monochromatic oscillations to nonstationary ones can be performed by means of inverse Fourier transformation, in accordance with the formula

$$B_i(x^1, x^2, x^3, t) = \int_{-\infty}^{\infty} \tilde{B}_i(x^1, x^2, x^3, \omega)\, e^{-i\omega t}\, d\omega.$$

We put

$$\mu_N(x^1, x^2, t) = \int_{-\infty}^{\infty} \tilde{\mu}_N(x^1, x^2, \omega)\, e^{-i\omega t}\, d\omega$$

$$= \oint e_N(x^1, x^3) \frac{\partial B_3(x^1, x^2, x^3, t)}{\partial x^2}\, dx^3. \quad (1)$$

From formulae (19), (24) and (27) of the previous paper we readily obtain

$$B_2(x^1, x^2, x^3, t) = \sum_N F_N(x^1, x^2, t) H_N(x^1, x^3), \quad (2)$$

where

$$F_N(x^1, x^2, t) = \int_{-\infty}^{\infty} \tilde{\mu}_N(x^1, x^2, \omega) \tilde{Q}_N(x^1, \omega)\, e^{-i\omega t}\, d\omega$$

$$= \int_{-\infty}^{\infty} \mu_N(x^1, x^2, t') Q_N(x^1, t-t')\, dt. \quad (3)$$

Here we have designated

$$Q_N(x^1, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{Q}_N(x^1, \omega)\, e^{-i\omega t}\, d\omega.$$

The function $Q_N(x^1, \tau)$ is as important in the theory of nonstationary oscillations as $\tilde{Q}_N(x^1, \omega)$ is in the theory of monochromatic oscillations. Basically, the problem of the time evolution of an Alfvén wave implies calculating $Q_N(x^1, \tau)$. This function has a number of properties associated with those of its Fourier-transform $\tilde{Q}_N(x^1, \omega)$. The function $Q_N(x^1, \tau)$ is real. Let us introduce the designations

$$Q_N^{(+)}(x^1, \tau) = \frac{1}{2\pi} \int_0^{\infty} \tilde{Q}_N(x^1, \omega)\, e^{-i\omega\tau}\, d\omega,$$

$$Q_N^{(-)}(x^1, \tau) = \frac{1}{2\pi} \int_{-\infty}^{0} \tilde{Q}_N(x^1, \omega)\, e^{-i\omega\tau}\, d\omega.$$

It is easy to see that $[Q_N^{(+)}(x^1, \tau)]^* = Q_N^{(-)}(x^1, \tau)$. This yields the equality

$$Q_N(x^1, t) = Q_N^{(+)}(x^1, t) + [Q_N^{(+)}(x^1, t)]^*,$$

from which it is evident that in order to determine Q_N, it is sufficient to calculate $Q_N^{(+)}$.

Analyticity of the function $\tilde{Q}_N(x^1, \omega)$ in the upper half-plane of the complex variable ω means that $Q_N(x^1, \tau) = 0$ when $\tau < 0$. Owing to this property, the causality principle

$$F_N(x^1, x^2, t) = \int_{-\infty}^{t} \mu_N(x^1, x_2, t') Q_N(x^1, t-t')\, dt'$$

$$= \int_0^{\infty} \mu_N(x^1, x^2, t-\tau) Q_N(x^1, \tau)\, d\tau \quad (4)$$

is satisfied; the value of the field at time t is determined by the source behaviour for the preceding period of time $t' < t$.

Let us consider several simple examples. For a monochromatic source with frequency ω_0, we have $\tilde{\mu}_N(x^1, x^2, \omega) = M_N(x^1, x^2)\delta(\omega - \omega_0)$. Then

$$F_N(x^1, x^2, t) = M_N(x^1, x^2) \tilde{Q}_N(x^1, \omega_0)\, e^{-i\omega_0 t}$$

$$= M_N(x_N^1(\omega_0), x^2) \tilde{Q}_N(x^1, \omega_0)\, e^{-i\omega_0 t}.$$

In the last equality it is assumed that a typical scale of variation of the function M_N in variable x^1 is much greater than that of the function $\tilde{Q}_N$. Thus, we again obtain the result we know from the theory of monochromatic oscillations (see Leonovich and Mazur, 1989), that the function $\tilde{Q}_N(x^1, \omega_0)$ describes the transverse structure of a monochromatic wave. If the source has the character of an instantaneous impulse $\mu_N(x^1, x^2, t) = m_N(x^1, x^2)\delta(t)$, then

$$F_N(x^1, x^2, t) = m_N(x^1, x^2) Q_N(x^1, t). \quad (5)$$

Thus, the function $Q_N(x^1, t)$ represents the response of Alfvén eigen-oscillations of the magnetosphere to an impulsive source. It is possible to generalize, to a significant extent, this result, and this is of practical interest in this case. We first remember that at a fixed value of x^1 the function $\tilde{Q}_N(x^1, \omega)$ is concentrated in the variable ω in the vicinity of points $\omega = \pm\Omega_N(x^1)$ on small intervals, whose typical size will be denoted as ω_N. Let the magnetosound wave be a time-bounded impulse, i.e., its amplitude is different from zero within the range $(0, \Delta t)$, with $\Delta t \ll \omega_N^{-1}$. A typical scale of variation of the Fourier-transform $\tilde{\mu}_N(x^1, x^2, \omega)$ in the variable ω, equal to $(\Delta t)^{-1}$, is much larger than ω_N. Therefore, from (3) it is easy to obtain

$$F_N(x^1, x^2, t) = |\tilde{\mu}_N(x^1, x^2, \Omega_N)|$$

$$\times [Q_N^{(+)}(x^1, t)\, e^{i\psi_N(x^1, x^2)} + Q_N^{(-)}(x^1, t)\, e^{-i\psi_N(x^1, x^2)}], \quad (6)$$

where $\psi_N(x^1, x^2) = \arg \tilde{\mu}_N(x^1, x^2, \Omega_N(x^1))$. Note that the condition $\Delta t \ll \omega_N^{-1}$ is a not too restrictive one because the value of ω_N^{-1} is quite large.

3. CALCULATING THE FUNCTION $Q_N(x^1, \tau)$

The expressions for $\tilde{Q}_N(x^1, \omega)$ which we obtained in our previous paper, permit us to define the function

$Q_N(x^1, \tau)$. Using formulae (29) and (30) from our previous paper, in the region of monotonic variation of the function $\Omega_N(x^1)$ we have

$$Q_N^{(+)}(x^1, \tau) = \frac{\Omega_N}{4\pi} e^{-i\Omega_N \tau} \int_{-\infty}^{\infty} \phi(\xi + i2^{1/3}\gamma_N \tau_N) \times \exp\left(-i\frac{\tau}{2^{1/3}\tau_N}\xi\right) d\xi = -\frac{i}{2}\Omega_N \theta(\tau) \times \exp(-i\Omega_N \tau - \gamma_N \tau - i\tau^3/6\tau_N^3).$$

Whence

$$Q_N(x^1, \tau) = -\Omega_N(x^1)\theta(\tau) e^{-\gamma_N(x^1)\tau} \times \sin[\Omega_N(x^1)\tau + \tau^3/6\tau_N^3]. \quad (7)$$

The relationships (2), (4) and (7) solve the problem of the evolution of the Alfvén wave field in the region of monotonic variation of the function $\Omega(x^1)$.

In formula (7) the transverse dispersion is involved in the term $\tau^3/6\tau_N^3$ in the sine argument. It begins to have effect when $\tau \gtrsim \tau_N$. In order that $Q_N(x^1, \tau)$ should then differ substantially from zero, it is necessary that the condition $\gamma_N \tau_N \lesssim 1$ we know from a previous paper be satisfied.

A clear interpretation may be given to the expression (7). As follows from the example given above, $Q_N(x^1, t)$ can be regarded as a response of the Alfvén wave field to a source of the type of an impulse which is instantaneous in time and constant in coordinates: $\mu_N(x^1, x^2, t) = \delta(t)$. From (5) it is evident that such a source produces a disturbance $F_N = Q_N(x^1, t)$ which when $t > 0$ is evolving freely. It may be treated as a set of a large number of one-dimensional wave packets which, at the initial moment of time, are uniformly distributed along the axis $x = x^1 - x_N^1$. The subsequent dynamics of the packets is defined by the equations

$$\frac{dx}{dt} = \frac{\partial \omega}{\partial K_x}, \quad \frac{dK_x}{dt} = -\frac{\partial \omega}{\partial x}. \quad (8)$$

Here $\omega = \omega(x, K_x)$ is a local frequency. For it, in the region of the monotonic variation of $\Omega_N(x^1)$, one can adopt the following model expression

$$\omega = \Omega_N(x^1)\left(1 + \frac{1}{2}K_x^2\rho_N^2\right) \approx \bar{\Omega}_N\left(1 - \frac{x}{l_N} + \frac{1}{2}K_x^2\rho_N^2\right). \quad (9)$$

From (8) and (9) it follows that the packet shows a uniformly accelerated motion. Let, when $t = 0$, the initial coordinate of the packet be $x = x_0$ and let the initial wave vector be $K_x = 0$ (this latter corresponds to smoothness of the initial disturbance). Then, at time t

$$x = x_0 - \bar{\Omega}_N^2 \rho_N^2 t^2 / 2l_N.$$

Equations (8) are known to retain the frequency $\omega(x, K_x)$ along the trajectory of the motion of the packet. This means that at time t at point x the field oscillations will proceed with such a frequency as at time $t = 0$ at point $x_0 = x + \bar{\Omega}_N^2 \rho_N^2 t^2/2l_N$, i.e., with the frequency

$$\tilde{\Omega}_N(x^1, t) = \Omega_N(x_0) = \Omega_N(x) + \frac{t^2}{2\tau_N^2}.$$

On the other hand, the expression (7) may be written as

$$Q_N(x^1, t) = -\Omega_N \theta(t) e^{-\gamma_N t} \sin\left[\int_0^t \tilde{\Omega}_N(x^1, t')\, dt'\right],$$

i.e., as the oscillations with variable frequency $\tilde{\Omega}_N(x, t)$.

In order to get a clear idea of the behaviour of Alfvén waves excited in the magnetosphere by a nonstationary magnetosound, we consider a specific example. We choose the source function $\mu_N(x^1, x^2, t)$ in the form

$$\mu_N(x^1, x^2, t) = M_N(x^1, x^2) e^{-\Gamma|t|} \sin \omega_0 t. \quad (10)$$

Such a source simulates a broad class of magnetospheric phenomena including type Pi2 pulsations and oscillations associated with SSC. Using for the function $Q_N(x^1, \tau)$ the expression (7) we can write (3) as

$$F_N(x^1, x^2, t) = -\frac{M_N(x^1, x^2)}{2}\left(\frac{l_N}{2\rho_N}\right)^{2/3} \times \int_0^{\infty} \exp[-\gamma_N \tau_N u - \Gamma \tau_N |u - u_0|] \times \left[\cos\left(\omega_0 t - (\omega_0 - \Omega_N)\tau_N u + \frac{u^3}{6}\right) - \cos\left(\omega_0 t - (\omega_0 + \Omega_N)\tau_N u - \frac{u^3}{6}\right)\right] du,$$

where $u_0 = t/\tau_N$. It will be assumed that $\gamma_N \ll \Gamma$ and $\Gamma\tau_N \gg 1$. Then, the expression under the integral has a sharp peak when $u = u_0$ and one may put $u^3 = u_0^3$ in the argument of cosines. After that, the integral is easily calculated to give

$$F_N(x^1, x^2, t) = -\frac{M_N \Omega_N^2}{(\Gamma^2 + (\omega_0 - \Omega_N)^2)(\Gamma^2 + (\omega_0 + \Omega_N)^2)} \times [(2\Gamma\omega_0 \cos\omega_0 t + (\omega_0^2 - \Omega_N^2 - \Gamma^2)\sin\omega_0 t) e^{-\Gamma|t|} + 4\Gamma\omega_0 \theta(t) e^{-\gamma_N t} \cos(\Omega_N t + t^3/6\tau_N^3)]. \quad (11)$$

The first term in square brackets represents induced

Alfvén oscillations associated with the direct action of magnetosound waves. Their lifetime is bounded by the time of action of the source and when $t \gg \Gamma^{-1}$ (but $t < \gamma_N^{-1}$) in the magnetosphere there remain only Alfvén-eigen modes described by the second term in the square bracket. Note that this last result can be obtained directly from formula (5) describing the response of the eigen-oscillations of the magnetosphere to the action of a short-duration source. As is apparent in (11), if $\gamma_N \ll \tau_N^{-1}$, then at $t \gtrsim \tau_N$ the magnetic shell under consideration shows a growth in frequency of the eigen-oscillations associated with dispersion effects. As far as the amplitude of the excited oscillations is concerned, it depends largely on the relation of frequencies ω_0 and Ω_N. The amplitude becomes largest when $|\omega_0 - \Omega_N| \ll \Gamma$, which corresponds to the resonance condition for induced and eigen-oscillations. When $\omega_0 \gg \Omega_N$ and $\omega_0 \ll \Omega_N$ the amplitude of waves is considerably smaller. Qualitatively, the oscillations excited by a source of the form (7) are depicted in Fig. 1.

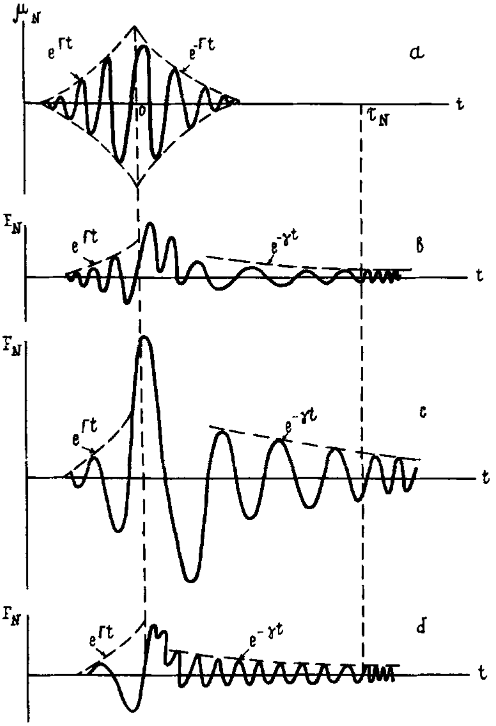

FIG. 1. THE TIME BEHAVIOUR OF STANDING ALFVÉN WAVES EXCITED IN THE MAGNETOSPHERE BY A NONSTATIONARY MAGNETOSOUND SOURCE OF THE FORM (a).
Shown are different cases: (b) the frequency of the excited wave, ω_0, is much higher than the frequency of Alfvén eigen-oscillations, Ω_N; (c) $|\omega_0 - \Omega_N| \ll \Gamma$ resonance of forcing and eigen-oscillations; and (d) $\omega_0 \ll \Omega_N$.

If the region of variation of x^1 considered here is located inside the dissipative layer, then formula (34) from a previous paper is valid for $\tilde{Q}_N(x^1, \omega)$. Using also the results presented in the Appendix to that paper we have

$$Q_N^{(+)}(x^1, \tau) = \frac{\Omega_N}{4\pi} \exp\left(-i\Omega_N \tau + i\frac{\alpha_N}{3}\right)$$
$$\times \int_{-\infty}^{\infty} \phi\,[e^{i\alpha_N/3}(\xi + i2^{1/3}\gamma_N \tau_N)] \exp\left(-i\frac{\tau\xi}{2^{1/3}\tau_N}\right) d\xi$$
$$= -\frac{i}{2}\Omega_N \theta(\tau) \exp\left(-i\Omega_N \tau - \gamma_N \tau - ie^{-i\alpha_N}\frac{\tau^3}{6\tau_N^3}\right).$$

Hence

$$Q_N(x^1, \tau) = -\Omega_N \theta(\tau) \exp\left(-\gamma_N \tau - \frac{\tau^3 \sin\alpha_N}{6\tau_N^3}\right)$$
$$\times \sin\left(\Omega_N \tau + \frac{\tau^3 \cos\alpha_N}{6\tau_N^3}\right). \quad (12)$$

This expression differs substantially from (7) by the presence of a term describing the super-exponential damping of the function $Q_N(x^1, \tau)$. The dissipative layer is notable for the value of $\sin\alpha_N \sim 1$. If $\gamma_N \ll 1$, then the damping time of the function $Q_N(x^1, \tau)$, on the order of magnitude, is τ_N. In this case integration in formula (4) of a source acting for a long time yields an Alfvén wave amplitude smaller than that outside the dissipative layer.

Near the extremum of the function $\Omega_N(x^1)$ the expression for $\tilde{Q}_N(x^1, \omega)$ is given by formula (35) from a previous paper. When calculating $Q_N(x^1, \tau)$, the integral over ω leads to the sum of residues at the poles $\omega = \pm\Omega_{Nn} - i\gamma_N$, which gives

$$Q_N(x^1, \tau) = -\bar{\Omega}_N \theta(\tau)\, e^{-\gamma_N \tau} \sum_{n=0}^{\infty} C_n y_n(x/\mathscr{C}) \sin(\Omega_{Nn}\tau).$$

Let us obtain for this series a closed expression. We rewrite it as

$$Q_N(x^1, \tau) = -\frac{i}{2}\bar{\Omega}_N \theta(\tau)\, e^{-\gamma_N \tau}\, [e^{-i\bar{\Omega}_N \tau} q(\zeta, \eta)$$
$$- e^{i\bar{\Omega}_N \tau} q^*(\zeta, \eta)], \quad (13)$$

where $\eta = \Delta\Omega_N \tau$, $\zeta = x/\mathscr{C}$, and the function $q(\zeta, \eta)$ is defined by the equality

$$q(\zeta, \eta) = \sum_{n=0}^{\infty} C_n y_n(\zeta) \exp(-i\lambda_n \eta/2).$$

It satisfies the following equation and initial condition

$$i\frac{\partial q}{\partial \eta} = -\frac{1}{2}\frac{\partial^2 q}{\partial \zeta^2} + \frac{\zeta^2}{2}q, \quad q(\zeta, 0) = 1.$$

The solution of the last problem is sought in a class of expressions $q = \exp(a\zeta^2 + \mathscr{C})$, where the functions $a = a(\eta)$ and $\mathscr{C} = \mathscr{C}(\eta)$ depend only on η. They satisfy simple differential equations, whose solutions are readily determined. As a result, we obtain

$$q(\zeta, \eta) = |\cos\eta|^{-1/2} \exp\left[-\frac{i}{2}\zeta^2 \tan\eta\right].$$

Substitution of this expression into (10) gives

$$Q_N(x^1, \tau) = -\bar{\Omega}_N \theta(\tau) e^{-\gamma_N \tau} |\cos\Delta\Omega_N \tau|^{-1/2} \times \sin[\bar{\Omega}_N \tau - (x^2/2\mathscr{C}^2) \tan\Delta\Omega_N \tau]. \quad (14)$$

The relationship (14) may also be given a clear interpretation in terms of wave packets. Near the maximum of our interest the dynamics of wave packets is determined by a local frequency

$$\omega = \bar{\Omega}_N (1 - x^2/2a_N^2 - K_x^2 \rho_N^2/2). \quad (15)$$

From (15) and (8) follows the law of motion of a wave packet with initial values $x = x_0$ and $K_x = 0$:

$$x = x_0 \cos\Delta\Omega_N t.$$

Thus, $\Delta\Omega_N$ has the sense of the oscillation frequency of a wave packet near the maximum $\Omega_N(x^1)$. By analogy with the above reasoning, for a variable oscillation frequency at point x, we have:

$$\tilde{\Omega}_N(x, t) = \Omega_N(x_0) = \bar{\Omega}_N\left(1 - \frac{x_0^2}{2a_N^2}\right) = \bar{\Omega}_N\left(1 - \frac{x^2}{2a_N^2 \cos^2\Delta\Omega_N t}\right).$$

This expression agrees with formula (14) which can be rewritten as

$$Q_N(x, \tau) = -\bar{\Omega}_N \theta(\tau) e^{-\gamma_N \tau} |\cos\Delta\Omega_N \tau|^{-1/2} \times \sin\left[\int_0^\tau \tilde{\Omega}_N(x, t)\, dt\right].$$

The pattern of motion of the wave packets suggests another, intriguing, conclusion. As a consequence of the independence of the oscillation period of a harmonic oscillator of the amplitude, all packets which are initially uniformly distributed along the x-axis, at time $\tau_0 = \pi/2\Delta\Omega_N$ as well as following any whole number of periods, will be located at point $x = 0$. This just explains the singularity of the function $Q_N(x, \tau)$ when $\tau = \tau_0$. Incidentally, this singularity is integrable and does not lead to complications in the relationship (4). Besides, if the oscillator's anharmonicity at finite values of x is taken into account, then the oscillation period of the packets will become dependent on amplitude and their simultaneous "incidence" on point $x = 0$ will not occur. As a result, the magnitude of the field at time $\tau = \tau_0$, though it increases greatly, remains a finite one.

Resonator properties of the maximum $\Omega_N(x^1)$ manifest themselves when $\gamma_N \ll \Delta\Omega_N$ [we have already obtained this result by considering the function $\tilde{Q}_N(x^1, \omega)$]. In the inverse case $\gamma_N \gg \Delta\Omega_N$ it suffices to limit attention to times $\tau \ll (\Delta\Omega_N)^{-1}$ because when $\tau \gtrsim (\Delta\Omega_N)^{-1}$ we have $\gamma_N \tau \gg 1$ and the expression (14) is virtually zero. Using the smallness of the value $\Delta\Omega_N \tau$ we obtain

$$Q_N(x^1, \tau) = -\bar{\Omega}_N \theta(\tau) e^{-\gamma_N \tau} \sin[\Omega_N(x^1)\tau - \tau^3/6\tau_N^3]. \quad (16)$$

Here

$$\Omega_N(x^1) = \bar{\Omega}_N(1 - x^2/2a_N^2), \quad \tau_N = (l_N/\rho_N)^{2/3} \bar{\Omega}_N^{-1}, \quad l_N^{-1} = x/a_N^2.$$

We want to stress that the last definition of the parameter l_N agrees with formula (31) from a paper of Leonovich and Mazur (1989). The difference of the expression (16) from (7) is caused by the different sign of dispersion. Thus, when $\gamma_N \gg \Delta\Omega_N$ relationships of the form (16) or (7) are also applicable near extrema of the function $\Omega(x^1)$. In the immediate vicinity of the extrema the dispersion is unimportant, but at a distance from them the parameter l_N increases; also where $\gamma_N \tau_N \gtrsim 1$ the dispersion plays its role.

4. EXCITATION OF ALFVÉN WAVES BY A STOCHASTIC SOURCE

In many practically important cases the magnetosound wave field has a stochastic character. Formula (3) in this case cannot be regarded as a final solution of the problem because the quantity $\mu_N(x^1, x^2, t)$ involved in it is a random function of time and cannot be considered given. Statistic characteristics of an ensemble of random functions describing a magnetosound wave should be considered given. We shall assume the ensemble to be a stationary one. For the present purposes it is sufficient to know a pair correlator of field $\tilde{B}_3(x^1, x^2, x^3, \omega)$:

$$\langle \tilde{B}_3^*(x^1, x^2, x^3, \omega) \tilde{B}_3(x^{1'}, x^{2'}, x^{3'}, \omega') \rangle = \mathscr{C}^*(x^1, x^2, x^3, \omega) \mathscr{C}(x^{1'}, x^{2'}, x^{3'}, \omega) \delta(\omega - \omega'). \quad (17)$$

An important feature in (17) is the statement about factorization (into two terms) of the correlation function.

In order to prove this statement, we must note that the Fourier-component $\tilde{B}_3(x^1, x^2, x^3, \omega)$ is a random function of frequency but a nonrandom function of coordinates. With a given frequency, it satisfies a certain equation with appropriate boundary conditions and matching conditions on resonance surfaces [more specifically, equation (8b) and the matching conditions (38) in a paper of Leonovich and Mazur (1989), in this issue]. Consequently, it is proportional to some standard, normalized in a certain manner, solution $\psi(x^1,x^2,x^3,\omega)$ of this equation:

$$\tilde{B}_3(x^1,x^2,x^3,\omega) = \eta(\omega)\psi(x^1,x^2,x^3,\omega), \quad (18)$$

where $\eta(\omega)$ is a random complex-valued function of frequency which specifies the amplitude and phase of a magnetosound harmonic. By virtue of statistic stationarity of the process its pair correlator has the form

$$\langle \eta(\omega)\eta(\omega')\rangle = f(\omega)\delta(\omega-\omega'). \quad (19)$$

The function $f(\omega)$ is real and non-negative. On denoting $\mathscr{C}(x^1,x^2,x^3,\omega) = f^{1/2}(\omega)\psi(x^1,x^2,x^3,\omega)$, from (18) and (19) we obtain the desired relationship (17).

Thus, the function $\mathscr{C}(x^1,x^2,x^3,\omega)$ at a given frequency describes the spatial dependence of the field of a monochromatic magnetosound wave. Its amplitude specifies the spectral composition of the field; in other words, the quantity $|\mathscr{C}(x^1,x^2,x^3,\omega)|^2$ can be regarded as a spectral function at point (x^1,x^2,x^3).

We shall confine ourselves to calculating two statistical characteristics of an Alfvén wave, namely a correlation function of two Fourier-harmonics at one point of the space and a mean square of the field amplitude. For the first one, we have

$$\langle \tilde{B}_2^*(x^1,x^2,x^3,\omega)\tilde{B}_2(x^1,x^2,x^3,\omega')\rangle = \sum_{N,N'} \times H_N(x^1,x^3)H_{N'}(x^1,x^3) \times \tilde{Q}_N^*(x^1,\omega)\tilde{Q}_{N'}(x^1,\omega) \times \tilde{\beta}_N^*(x^1,x^2,\omega)\tilde{\beta}_{N'}(x^1,x^2,\omega)\delta(\omega-\omega').$$

Here it is designated

$$\tilde{\beta}_N(x^1,x^2,\omega) = \oint e_N(x^1,x^2)\frac{\partial \mathscr{C}(x^1,x^2,x^3,\omega)}{\partial x^2}\,dx^3.$$

Let us consider the values $\omega > 0$. At a given x^1 the functions $\tilde{Q}_N(x^1,\omega)$ and $\tilde{Q}_{N'}(x^1,\omega)$ are localized in variable ω in the narrow vicinities of points $\omega = \Omega_N(x^1)$ and $\omega = \Omega_{N'}(x^1)$, respectively. Therefore, their product can be assumed equal to zero if $N \neq N'$. For the same reason, one can consider the dependence of $\tilde{\beta}_N(x^1,x^2,\omega)$ on ω to be much smoother than $\tilde{Q}_N(x^1,\omega)$ and can put

$$\tilde{\beta}_N(x^1,x^2,\omega)\tilde{Q}_N(x^1,\omega) \approx \tilde{\beta}_N(x^1,x^2,\Omega_N(x^1))\tilde{Q}_N(x^1,\omega).$$

As a result, we obtain

$$\langle \tilde{B}_2^*(x^1,x^2,x^3,\omega)\tilde{B}_2(x^1,x^2,x^3,\omega')\rangle = P(x^1,x^2,x^3,\omega)\delta(\omega-\omega'),$$

where

$$P(x^1,x^2,x^3,\omega) = \sum_N H_N^2(x^1,x^3) \times |\tilde{\beta}_N(x^1,x^2,\Omega_N(x^1))|^2|\tilde{Q}_N(x^1,\omega)|^2. \quad (20)$$

This quantity has the meaning of a spectral function of Alfvén oscillations.

The obtained result allows us to easily calculate the mean square of the field amplitude of an Alfvén wave. We have

$$\langle B_2^2(x^1,x^2,x^3,t)\rangle = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \times \langle \tilde{B}_2^*(x^1,x^2,x^3,\omega)\tilde{B}_2(x^1,x^2,x^3,\omega')\rangle\, e^{i(\omega-\omega')t} = 2\int_0^{\infty} P(x^1,x^2,x^3,\omega)\,d\omega.$$

We shall assume that formula (34) from a paper of Leonovich and Mazur (1989) is applicable for the values of x^1 of interest. Then, using results of the Appendix in that paper we obtain

$$\int_0^{\infty} |\tilde{Q}_N(x^1,\omega)|^2 d\omega = \frac{\Omega_N}{2}\left(\frac{l_N}{2\rho_N}\right)^{2/3}\int_{-\infty}^{\infty} \times |\phi[e^{i\alpha_N/3}(\xi + i2^{1/3}\gamma_N\tau_N)]|^2 d\xi = \frac{\pi}{2^{2/3}} \times \frac{\Omega_N^2\tau_N}{(\sin\alpha_N)^{1/3}}\psi\left[\frac{2\gamma_N\tau_N}{(\sin\alpha_N)^{1/3}}\right].$$

Therefore

$$\langle B_2^2(x^1,x^2,x^3,t)\rangle = 2^{1/3}\pi\sum_N H_N^2(x^1,x^3) \times |\tilde{\beta}_N(x^1,x^2,\Omega_N(x^1))|^2\frac{\Omega_N^2\tau_N}{(\sin\alpha_N)^{1/3}} \times \psi\left[\frac{2\gamma_N\tau_N}{(\sin\alpha_N)^{1/3}}\right]. \quad (21)$$

If $\gamma_N\tau_N \gg (\sin\alpha_N)^{1/3}$ for all harmonics N of interest, then we have

$$\langle B_2^2(x^1,x^2,x^3,t)\rangle = \frac{\pi}{2^{2/3}}\sum_N \frac{\Omega_N^2(x^1)}{\gamma_N(x^1)} \times H_N^2(x^1,x^3)|\tilde{\beta}_N(x^1,x^2,\Omega_N(x^1))|^2. \quad (22)$$

In the inverse limit $\gamma_N\tau_N \ll (\sin\alpha_N)^{1/3}$

$$\langle B_2^2(x^1,x^2,x^3,t)\rangle = \pi 6^{1/3}\Gamma\left(\frac{1}{3}\right)\sum_N \frac{\Omega_N^2\tau_N}{(\sin\alpha_N)^{1/3}} \times H_N^2(x^1,x^3)\,|\tilde{\beta}_N(x^1,x^2,\Omega_N(x^1))|^2. \quad (23)$$

Formulae (20)–(23) resolve the question of the distribution in the magnetosphere of the spectrum and amplitude of Alfvén oscillations excited by a stochastic stationary source.

If for numbers N of our interest the inequality $\gamma_N\tau_N \gg 1$ is satisfied, then the expression (22) is valid, which describes in this case the field amplitude both outside and inside the dissipative layer. Of more interest is the inverse limit $\gamma_N\tau_N \ll 1$, when outside the dissipative layer where $\sin\alpha_N = 0$, formula (23) is again applicable, and inside it when $\sin\alpha_N \gg (\gamma_N\tau_N)^3$, formula (23) applies. It is remarkable that despite the inequality $\gamma_N\tau_N \ll 1$ the dependence on the dispersion parameter vanishes in formula (22). This fact can be given the following explanation. Transverse dispersion causes the wave to move across the magnetic shells and, therefore, leads to energy escape from the initial shell. But for a broadband source the neighbouring shells are also filled with oscillations and the energy input there compensates for the output. In other words, the real dissipation is determined only by damping in the ionosphere, which is just reflected in (22). The presence of the dispersion parameter τ_N in (23) is due not to the dispersion as such but to the fact that wave damping on electrons in the dissipative layer is associated with this parameter.

As has already been pointed out, an extensive interpretation of the different types of geomagnetic pulsations on the basis of the formulae obtained is beyond the scope of the present study; however, we shall briefly discuss one phenomenon. The question is concerned with an important class of daytime Pc3 pulsations. A review of Gul'elmi (1984) presents evidence that their source is an instability in a flux of protons reflected from the front of a bow shock. The instability generates a magnetosound wave which then penetrates into the magnetosphere. Using a model of a stochastic stationary source for interpretation of this phenomenon appears to be quite justifiable.

An important role in the formulae derived above is played by the function $\tilde{\beta}_N(x^1,x^2,\omega)$ which is determined by the field of magnetosound. We shall consider the magnetosphere for it to be an opacity region. This means that $\tilde{\beta}_N(x^1,x^2,\omega)$ as a function of x^1 drops monotonically into the magnetosphere. As a function of ω it has a maximum at the frequency $\bar{f} = \bar{\omega}/2\pi$ where the wave excitation mechanism is the most effective. This is illustrated in Fig. 2, in its upper part on the left. Plots of the functions $f_N(x^1) = \Omega_N(x^1)/2\pi$ are shown on the right. The horizontal bar corresponds to the range of frequencies excited by the instability. Estimates show that the central frequency $\bar{f}$ coincides roughly with the value of the minimum of the function $\Omega_1(x^1)/2\pi$ on the inner edge of plasmapause. Using this figure, one can gain a qualitative idea of the spatial behaviour of the functions $\tilde{\beta}_N(x^1,x^2,\Omega_N(x^1))$ for $x^2 = \text{const}$. Relevant schematic plots are shown in the lower part of Fig. 2. Two maxima of the function $\tilde{\beta}_1(x^1,x^2,\Omega_N(x^1))$ in variable x^1 are due to the fact that the horizontal band in two places intersects curve $\Omega_1(x^1)$, i.e., the last argument of the function $\tilde{\beta}_1(x^1,x^2,\omega)$ in two places affords a maximum for it. The left-hand maximum lies below the right-hand maximum because of the drop of the function $\tilde{\beta}_1(x^1,x^2,\omega)$ inward to the magnetosphere in the first argument. The heights of maxima of the functions, $\tilde{\beta}_2, \tilde{\beta}_3, \ldots$ decrease because of the increase of number N.

Let us turn now to formulae (20)–(23). The first of them describes the pulsation spectrum. The presence in it of the terms $|\tilde{Q}_N(x^1,\omega)|^2$ means that the spectrum consists of narrow peaks near frequencies $\omega = \Omega_N(x^1)$ of width $\Delta\omega = \tau_N^{-1}$. This conclusion agrees well with

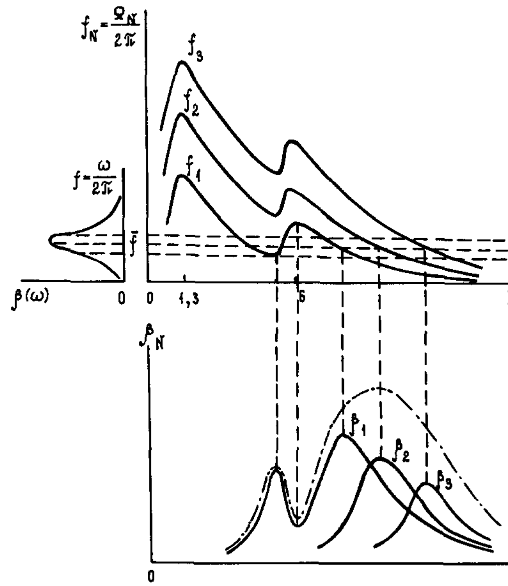

FIG. 2. THE UPPER PLOT SHOWS THE DEPENDENCES OF FREQUENCIES OF THE THREE FIRST HARMONICS OF THE EIGENOSCILLATIONS OF THE MAGNETOSPHERE f_N ON THE PARAMETER L AND A SPECTRAL FUNCTION OF THE SOURCE OF FMS WAVES $\beta(\omega)$ (LEFT).
The lower plot shows amplitude profiles of appropriate harmonics β_N for a given spectral function $\beta(\omega)$ and the total amplitude profile (broken line).

satellite observations, both geostationary (Takahashi and McPherron, 1984) and those orbiting with large eccentricity (Engebretson *et al.*, 1986). The heights of spectral peaks are determined mainly by the functions $|\tilde{\beta}_N(x^1, x^2, \Omega_N(x^1))|^2$. This means that the inner part of the magnetosphere must be dominated by a peak $\Omega_1(x^1)$ and the outer part must be dominated by $\Omega_2(x^1), \Omega_3(x^1)$, etc., that is, the eigen-frequency which is roughly equal to frequency $\bar{\omega} = 2\pi\bar{f}$. This last fact makes ground-based observations understandable. The presence of the ionosphere and atmosphere "washes out" the harmonic structure of the spectrum with the result that the dominant oscillation frequency proves to be the same at all latitudes (i.e., $\bar{f}$).

Formulae (22) and (23) provide insights into the meridional profile by Pc3 amplitude. It is also determined mainly by the functions $|\tilde{\beta}_N(x^1, x^2, \Omega_N(x^1))|^2$. The term with $N = 1$ gives two maxima, one for mid-latitudes and the other for high latitudes, with a significant valley between them lying approximately on the outer edge of the plasmapause. The second maximum is larger than the first one. Terms with $N = 2, 3, \ldots$ somewhat expand and increase the high-latitude maximum. The term Ω_N^2/γ_n does not make a clear contribution of its own but, most likely, cannot alter the overall picture drastically. As far as the dissipative layer is concerned, the decrease of amplitude in it is able to further deepen the minimum when its position coincides with the plasmapause or may produce another minimum in a different place. The theoretical picture of the behaviour of the pulsation amplitude described here agrees with main ground-based observations of the amplitude profile of Pc3 (Pudovkin *et al.*, 1976).

5. CONCLUSIONS

Let us formulate the main results of this paper.

(1) We have obtained general formulae which define the space-and-time behaviour of the field of standing Alfvén waves in an axisymmetric magnetosphere. Alfvén oscillations are produced by a magnetosound wave penetrating from outside into the magnetosphere, with an arbitrary behaviour in time. The formulae include the dispersion effect of Alfvén waves as well as the effect of their dissipation in the ionosphere. They are obtained using an inverse Fourier-transform of formulae of the theory of monochromatic waves as developed in our previous paper (Leonovich and Mazur, 1989).

(2) General relationships involve the function $Q_N(x^1, t)$ which is the response of the Nth longitudinal harmonic of the Alfvén wave field to an instantaneous impulse. This function has been calculated for all physically different cases of the behaviour of magnetospheric parameters, namely in the region of monotonic variation of the function $\Omega_N(x^1)$, near its extrema, and also in the dissipative layer in which the Cherenkov damping of the waves due to electrons is important. As a result, the general relationships acquire a constructive content. We have considered an example which is useful for modelling the Pi2 pulsation, SSC, and other disturbances excited by a short-duration impulse.

(3) A study has been made of the excitation of Alfvén waves by a stochastic magnetosound which can be generated by a variety of instabilities. The spatial distribution of the spectrum and amplitude of Alfvén oscillations has been determined. As an example of application of the formulae obtained we have considered geomagnetic Pc3 pulsations. It has been shown that the theory makes it possible to explain the main features of the observational picture, i.e., the spatial distribution of the spectrum and amplitude of the oscillations.

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