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The turbulent stresses in the theory of the solar torsional oscillations

G. Rüdiger[1] **and L.L. Kichatinov**[2]

[1] Sternwarte Babelsberg, DDR-1591 Potsdam, German Democratic Republic
[2] Sibirian Institute of Terrestrial Magnetism, Irkutsk 33, P.O. Box 4, USSR

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Abstract. We consider the influence of a large-scale magnetic field, $\langle \boldsymbol{B} \rangle$, on both the Reynolds and turbulent Maxwellian stresses. In the frame of the Second-Order-Correlation-Approximation we determine this general influence on the Maxwellian stress $\langle B_i' B_j' \rangle$ and on the turbulent angular momentum transport (TMT). We find the turbulent Maxwellian stress to act opposite to the mean-field constituent $\langle B_i \rangle \langle B_j \rangle$, hence the general tensor $\langle B_i B_j \rangle$ may degenerate for the often discussed ("Orszag's") turbulence models to a pure diagonal (pressure) form – or, what is the same, the averaged Lorentz force $\langle \boldsymbol{j} \times \boldsymbol{B} \rangle$ is conservative. This surprising result provides an alternative to the hitherto sole explanation of the solar "torsional oscillation" as being Lorentz-force-induced modification of the mean-flow system in the convection zone. In addition, for stellar applications the influence of $\langle \boldsymbol{B} \rangle$ on the TMT-expressions for anisotropic turbulence proves to be much stronger than the normal large-scale Lorentz force. This "Lambda-effect quenching" is thus proposed to be the essential generator of the cyclic contributions to the Sun's differential rotation.

Key words: MHD – turbulence – Sun: torsional oscillations

1. Introduction

In a first approximation the convective envelopes of cool stars can be considered as rotating electrically conducting turbulence under the presence of a large-scale magnetic field. It was the mean-field dynamo theory which revealed the interplay between the general rotation and the characteristics of the turbulence for the generation of the overall stellar magnetism – which by itself forms the physical background for essential parts of the highly detailed phenomena of "stellar activity". Besides the often discussed α-effect the non-uniformity of the basic rotation yields the other input parameter to the MHD dynamo equations. This differential rotation can also be understood as a turbulence effect in rotating media, more exactly speaking, as a consequence of its internal angular momentum transport. The flux of angular momentum in turbulent media no longer vanishes for solid-body rotation if the turbulence is anisotropic. In this case, even for Ω = const. non-diffusive finite Reynolds stresses exist,

$$Q_{ij} = \Lambda_{ijk} \Omega_k, \tag{1}$$

which in interaction with the eddy viscosity (i.e. diffusive-) fluxes allows only non-uniform rotation to be a solution of the Navier-Stokes equation. This so-called Λ-effect, (1), in its *close relation to the turbulence theory*, has previously been discussed by Biermann (1951), Elsässer (1966), Rüdiger (1977), Gough (1978), Durney and Spruit (1979), and Kichatinov (1986). The current status and history of the general hydrodynamic theory of differential rotation is presented in the monograph by Rüdiger (1989).

What is still open in this scenario is the feedback of the dynamo-generated magnetic fields on the turbulent angular momentum transport (TMT), i.e. the influence of the large-scale magnetic field $\langle \boldsymbol{B} \rangle$ on the resulting differential rotation. We generally expect this feedback to take the form of a reduction factor for the non-uniformity of the angular velocity $\boldsymbol{\Omega}$. Hence it gives one of the basic nonlinearities of the stellar dynamo (Malkus and Proctor, 1975). In particular, the relation between this term and the influence of the mean-field Lorentz force $\langle \boldsymbol{j} \rangle \times \langle \boldsymbol{B} \rangle$ as the source of the observed solar "torsional oscillation" is of interest in this connection.

2. Basic equations

The equations for the fluctuations of momentum and magnetic field are well-known. We apply them in a rotating frame of reference in which the density as well as the mean magnetic field $\langle \boldsymbol{B} \rangle$ are assumed to be *homogeneous*. Then the linearized equations for the random parts of flow and field are respectively

$$\partial \boldsymbol{u}'/\partial t - \nu \Delta \boldsymbol{u}' = -\frac{1}{\rho} \nabla p' + \frac{1}{\rho \mu} \operatorname{curl} \boldsymbol{B}' \times \langle \boldsymbol{B} \rangle - 2\boldsymbol{\Omega} \times \boldsymbol{u}' + \boldsymbol{f}',$$

$$\partial \boldsymbol{B}'/\partial t - \eta \Delta \boldsymbol{B}' = \operatorname{curl}(\boldsymbol{u}' \times \langle \boldsymbol{B} \rangle), \tag{2}$$

with

$$\operatorname{div} \boldsymbol{u}' = \operatorname{div} \boldsymbol{B}' = 0. \tag{3}$$

As usual in such calculations the *external* acceleration field $\boldsymbol{f}'$ maintains the turbulence which otherwise would rapidly decay. As similar outlines have already been presented elsewhere we can restrict ourselves to a short account of the mathematical manipulations yielding the desired results.

Send offprint requests to: G. Rüdiger

Let us introduce the Alfvén velocity vector $\boldsymbol{v} \equiv \boldsymbol{B}/(\mu\rho)^{1/2}$ and also for both velocity fluctuation vectors their Fourier transforms

$$(\boldsymbol{u}', \boldsymbol{v}') = \int (\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}) e^{i(\boldsymbol{kx}-\omega t)} \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega, \tag{4}$$

with similar transformation for pressure and force fluctuations.

Then the Fourier transformed Eqs. (2) and (3) take the form

$$(-i\omega + \nu k^2)\hat{u}_i + 2\varepsilon_{ijk}\Omega_j \hat{u}_k - i((\boldsymbol{kv})\hat{v}_i - (\boldsymbol{v}\hat{\boldsymbol{v}})k_i) = -ik_i\hat{p}/\rho + \hat{f}_i,$$
$$(-i\omega + \eta k^2)\hat{v}_i = i(\boldsymbol{kv})\hat{u}_i, \quad \boldsymbol{k}\hat{\boldsymbol{u}} = \boldsymbol{k}\hat{\boldsymbol{v}} = 0, \tag{5}$$

where from now on $\boldsymbol{v}$ denotes the statistical average $\langle \boldsymbol{v} \rangle$. Eliminating the pressure terms via multiplication with

$$E_{ij} = \delta_{ij} - \dot{k}_i \dot{k}_j, \quad \dot{\boldsymbol{k}} \equiv \boldsymbol{k}/k, \tag{6}$$

and the magnetic fluctuations in $(5)_1$, we arrive at the equation

$$\{-i\omega + \nu k^2 + (\boldsymbol{kv})^2/(-i\omega + \eta k^2)\}\hat{\boldsymbol{u}} + (2\dot{\boldsymbol{k}}\boldsymbol{\Omega})(\dot{\boldsymbol{k}} \times \hat{\boldsymbol{u}}) = \hat{\boldsymbol{f}} - (\dot{\boldsymbol{k}}\hat{\boldsymbol{f}})\dot{\boldsymbol{k}}, \tag{7}$$

which for the two separate cases $\Omega = 0$ and $\boldsymbol{B} = 0$ has already been discussed in previous papers (Rüdiger, 1974; Rüdiger, 1977).

Typically we proceed as follows. As the RHS of Eq. (7) is *assumed* to be independent of any large-scale parameter we can there put $\boldsymbol{\Omega} = \boldsymbol{v} = 0$ and find

$$(-i\omega + \nu k^2)\hat{\boldsymbol{u}}^{(0)} = \hat{\boldsymbol{f}} - (\dot{\boldsymbol{k}}\hat{\boldsymbol{f}})\dot{\boldsymbol{k}}, \tag{8}$$

with $\boldsymbol{u}^{(0)}$ as the so-called *original* turbulence which is thought to exist in the same way also in non-rotating and non-magnetic stars. One must understand in this sense the following basic relation

$$\{-i\omega + \nu k^2 + (\boldsymbol{kv})^2/(-i\omega + \eta k^2)\}\hat{\boldsymbol{u}} + (2\dot{\boldsymbol{k}}\boldsymbol{\Omega})(\dot{\boldsymbol{k}} \times \hat{\boldsymbol{u}})$$
$$= (-i\omega + \nu k^2)\hat{\boldsymbol{u}}^{(0)}$$

as a special interpretation of (7).

We express the solution of this equation as

$$\hat{u}_i = L_{im}\hat{u}_m^{(0)}, \tag{10}$$

and find after some algebra the general result

$$L_{im} = (-i\omega + \nu k^2)(-i\omega + \eta k^2)[\{(-i\omega + \nu k^2)(-i\omega + \eta k^2) + (\boldsymbol{kv})^2\}\delta_{im} + (-i\omega + \eta k^2)(2\dot{\boldsymbol{k}}\boldsymbol{\Omega})\varepsilon_{imp}\dot{k}_p]/N \tag{11}$$

with

$$N = ((-i\omega + \nu k^2)(-i\omega + \eta k^2) + (\boldsymbol{kv})^2)^2 + (-i\omega + \eta k^2)^2(2\dot{\boldsymbol{k}}\boldsymbol{\Omega})^2. \tag{12}$$

Note the difference between the influence of $\boldsymbol{B}$ and Ω. Only at one place qualitatively new ("odd") terms appear while the other "even" terms in $\boldsymbol{B}$ and Ω only modify the existing turbulent spectrum.

The solution of $(5)_2$ may be formulated in quite a similar way:

$$\hat{\boldsymbol{v}} = L\hat{\boldsymbol{u}}$$

with

$$L = i(\boldsymbol{kv})/(-i\omega + \eta k^2). \tag{13}$$

We now restrict ourselves to the consideration of *homogeneous* turbulence which results in great mathematical simplifications. Only in this case it is possible to proceed directly from (10) and $(13)_1$ to the spectral tensors

$$\hat{\mathbf{Q}}_{ij} = L^*_{im}L_{jn}\hat{Q}^{(0)}_{mn}$$

and

$$\hat{\mathbf{Q}}^{\mathrm{mag}}_{ij} = L^* L\hat{Q}_{ij}, \tag{14}$$

the latter for the correlations $\langle v'_i v'_j \rangle$. The totality of both Reynolds and Maxwellian stress,

$$\mathbf{Q}^{\mathrm{tot}} = \mathbf{Q} - \mathbf{Q}^{\mathrm{mag}}, \tag{15a}$$

in Fourier space becomes

$$\hat{\mathbf{Q}}^{\mathrm{tot}}_{ij} = \hat{\mathbf{Q}}_{ij} - \hat{\mathbf{Q}}^{\mathrm{mag}}_{ij} = (1 - (\boldsymbol{kv})^2/(\omega^2 + \eta^2 k^4))\hat{\mathbf{Q}}_{ij}. \tag{15b}$$

Only this tensor represents the total influence of small-scale fluctuations on the mean-field quantities.

It prove to be worthwhile to split the spectral tensor into two parts,

$$\mathbf{Q} = \mathbf{Q}^{\mathrm{even}} + \mathbf{Q}^{\mathrm{odd}}, \tag{16}$$

with the numbers of the Ω's as the separation criterion. Due to its symmetry properties the odd part contains the covariance correlation $\mathbf{Q}_{r\varphi}$ as well as $\mathbf{Q}_{\theta\varphi}$, which are basic for the above mentioned TMT. It is in particular this odd part of the correlation tensors which shall be discussed in Sect. 4 of the present paper. Its general structure is

$$\hat{\mathbf{Q}}^{\mathrm{odd}}_{ij} = (\omega^2 + \nu^2 k^4)(\omega^2 + \eta^2 k^4)\{\nu(\omega^2 + \eta^2 k^4) + \eta(\boldsymbol{kv})^2\}(2\boldsymbol{k}\boldsymbol{\Omega})k_m(\varepsilon_{jlm}\hat{\mathbf{Q}}^{(0)}_{il} + \varepsilon_{ilm}\hat{\mathbf{Q}}^{(0)}_{lj})/NN^*, \tag{17}$$

if only the real contributions are considered. For the denominator the general expression

$$NN^* = \{(\nu\eta k^4 - \omega^2 + (\boldsymbol{kv})^2)^2 - \omega^2(\nu + \eta)^2 k^4 + (\eta^2 k^4 - \omega^2)(2\dot{\boldsymbol{k}}\boldsymbol{\Omega})^2\}^2 + 4\omega^2\{\eta k^2(2\dot{\boldsymbol{k}}\boldsymbol{\Omega})^2 + (\nu + \eta)k^2(\eta\nu k^4 - \omega^2 + (\boldsymbol{kv})^2)\}^2. \tag{18}$$

follows. For slow rotation and weak magnetic field a series expansion is possible and yields

$$NN^* \approx (\omega^2 + \nu^2 k^4)^2(\omega^2 + \eta^2 k^4)^2(1 + 4\beta(\boldsymbol{kv})^2) \tag{19}$$

with

$$\beta = (\nu\eta k^4 - \omega^2)/((\omega^2 + \nu^2 k^4)(\omega^2 + \eta^2 k^4)). \tag{20}$$

For vanishing diffusivities the denominator NN^* represents various interesting free solutions in rotating magnetized media. It can easily be shown that

$$NN^* = (\omega - \omega_1)^2(\omega - \omega_2)^2(\omega - \omega_3)^2(\omega - \omega_4)^2 \tag{21}$$

with the four eigenfrequencies

$$\omega_{1,\ldots,4} = \pm(\dot{\boldsymbol{k}}\boldsymbol{\Omega}) \pm ((\dot{\boldsymbol{k}}\boldsymbol{\Omega})^2 + (\boldsymbol{kv})^2)^{1/2}, \tag{22}$$

[cf. Acheson and Hide, 1973, their Eq. (4.11)] describing dispersive "hydromagnetic inertial waves".

3. The Maxwellian stress

Let us first analyze the even part of the tensor (16),

$$\hat{\mathbf{Q}}^{\mathrm{even}}_{ij} = (\omega^2 + \nu^2 k^4)(\omega^2 + \eta^2 k^4)[\{(\omega^2 + \eta^2 k^4)(\omega^2 + \nu^2 k^4 - (2\dot{\boldsymbol{k}}\boldsymbol{\Omega})^2) + 2(\boldsymbol{kv})^2(\nu\eta k^4 - \omega^2) + (\boldsymbol{kv})^4\}\hat{\mathbf{Q}}^{(0)}_{ij} + (\omega^2 + \eta^2 k^4)(2\dot{\boldsymbol{k}}\boldsymbol{\Omega})^2 E_{ij}\hat{\mathbf{Q}}^{(0)}_{ll}]/NN^*, \tag{23}$$

from which the positive-definite expression for the turbulent energy

$$\hat{\mathbf{Q}}_{ll}=(\omega^2+\nu^2k^4)(\omega^2+\eta^2k^4)\{(\omega^2+\eta^2k^4)(\omega^2+\nu^2k^4+(2\boldsymbol{k}\boldsymbol{\Omega})^2)+2(\nu\eta k^4-\omega^2)(\boldsymbol{k}\boldsymbol{v})^2+(\boldsymbol{k}\boldsymbol{v})^4\}\hat{\mathbf{Q}}_{ll}^{(0)}/NN^* \quad (24)$$

results (cf. Bräuer and Krause, 1979). It might be interesting to discuss the concerted action of rotation and magnetic field on turbulent media in more detail but this is beyond the scope of this paper. We restrict ourselves to a special aspect of this subject, i.e. the Maxwell stress. We want to find the influence of the macroscopic magnetic field $\langle \boldsymbol{B}\rangle$ on the macroscopic motion $\langle u\rangle$ via microscopic fluctuations. This part of Lorentz force has so far always been neglected in mean-field solar magnetohydrodynamics.

For a demonstration of the overall problem we cancel the rotational terms ($\boldsymbol{\Omega}=0$) in (23) and ignore magnetic terms of higher than second order:

$$\hat{\mathbf{Q}}_{ij}\approx(1-2\beta(\boldsymbol{k}\boldsymbol{v})^2)\hat{\mathbf{Q}}_{ij}^{(0)}. \quad (25)$$

For the total turbulence stress tensor (15) we obtain therefore

$$\hat{\mathbf{Q}}_{ij}^{\rm tot}=\{1-(2\beta+1/(\omega^2+\eta^2k^4))(\boldsymbol{k}\boldsymbol{v})^2\}\hat{\mathbf{Q}}_{ij}^{(0)}, \quad (26)$$

hence

$$\mathbf{Q}_{ij}^{\rm tot}=\mathbf{Q}_{ij}^{(0)}-\int\frac{2\nu\eta k^4+\nu^2k^4-\omega^2}{(\omega^2+\nu^2k^4)(\omega^2+\eta^2k^4)}(\boldsymbol{k}\boldsymbol{v})^2\hat{\mathbf{Q}}_{ij}^{(0)}\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega. \quad (27)$$

For homogeneous and isotropic turbulence, which cannot provide for a new characteristic direction, one can write

$$\mathbf{Q}_{ij}^{\rm tot}=-\kappa'\delta_{ij}+\kappa v_i v_j, \quad (28)$$

where the coefficient κ proves to be

$$\kappa=\frac{1}{15}\int\frac{\nu^2k^4+2\nu\eta k^4-\omega^2}{(\omega^2+\nu^2k^4)(\omega^2+\eta^2k^4)}k^2\hat{\mathbf{Q}}_{ll}^{(0)}\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega. \quad (29)$$

[cf. Roberts and Soward, 1975, Eq. (4.15)].

One can show that for many spectral functions the expression (29) is positive. It is certainly positive for $\hat{\mathbf{Q}}^{(0)}\approx\delta(\omega)$ (very long correlations) and it is also positive for $\partial\hat{\mathbf{Q}}^{(0)}/\partial\omega=0$ (white noise). One can write, moreover, for $\nu=\eta$,

$$\kappa=\frac{1}{15}\int\left\{\frac{2\nu^2k^6\hat{\mathbf{Q}}_{ll}^{(0)}}{(\omega^2+\nu^2k^4)^2}-\frac{\omega}{\omega^2+\nu^2k^4}\frac{\partial}{\partial\omega}(k^2\hat{\mathbf{Q}}_{ll}^{(0)})\right\}\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega, \quad (30a)$$

which is positive-definite for all spectral functions of turbulence-type, i.e. monotonically decreasing for increasing frequency.

Also interesting in this connection is the limit of high conductivity, $\eta\to 0$. For low magnetic Prandtl number (the solar value in the convection zone is $\mathrm{Pr}_m\approx 0.1$) this limit makes only sense, however, if first $\nu\to 0$. In this limit one finds

$$\kappa=\frac{2\pi}{15\eta}\int\hat{\mathbf{Q}}_{ll}^{(0)}(k,0)\mathrm{d}\boldsymbol{k}-\frac{1}{15}\int\frac{k^2\hat{\mathbf{Q}}_{ll}^{(0)}(k,\omega)}{\omega^2+\eta^2k^4}\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega, \quad (30b)$$

which reveals the basic features of low-diffusivity approaches. The extra limit $\eta\to 0$ is only possible for vanishing $Q(k,0)$:

$$\lim_{\eta\to 0}\eta\kappa=\frac{\pi}{15}\int\hat{\mathbf{Q}}_{ll}^{(0)}(k,0)\mathrm{d}\boldsymbol{k}. \quad (30c)$$

On the other hand, finite values of the magnetic diffusivity would lead to $\kappa\approx\tau\langle \boldsymbol{u}'^2\rangle/\eta$ (η_T/η in the language of Roberts and Soward, 1975) which for the Sun might reach values of order 10^9 (Stix, 1989, p. 253).

The adoption of the Vainshtein-Kichatinov approach [cf. Eq. (53) below] seems to be more realistic. It yields the positive-definite result

$$\kappa=\frac{\tau}{5}\int k^2\hat{\mathbf{Q}}_{ll}^{(0)}(k)\mathrm{d}\boldsymbol{k}, \quad (30d)$$

which itself, for Strouhal number of order unity, it also of order unity.

The result $\kappa\approx 1$ may be very relevant from the following reason. The complete stress tensor

$$\mathbf{T}_{ij}=\rho\{\langle u_i\rangle\langle u_j\rangle-\langle v_i\rangle\langle v_j\rangle\}+p\delta_{ij}+\rho\mathbf{Q}_{ij}^{\rm tot}+\dots \quad (31)$$

takes with (28) the structure

$$\mathbf{T}_{ij}=\rho\{\langle u_i\rangle\langle u_j\rangle-(1-\kappa)\langle v_i\rangle\langle v_j\rangle\}+p\delta_{ij}+\dots, \quad (32)$$

(cf. Roberts and Soward, 1975), so that for $\kappa\approx 1$ the magnetic influence is basically reduced (Rüdiger et al., 1986). Magnetic pressure terms ("κ'") are not explicitly given here as they do not at all influence the argument.

Obviously, the microscopic Lorentz force weakens the macroscopic one in such a way that we do not even know the resulting sign. For wide classes of turbulence the *off-diagonal components of the sum* $\langle B_iB_j\rangle\equiv\langle B_i\rangle\langle B_j\rangle+\langle B_i'B_j'\rangle$ may nearly vanish due to the influence of its second term. In the light of this statement we might even re-discuss the traditional explanation of the torsional oscillations of the Sun which are always considered as exclusively due to the macroscopic Lorentz force $\langle \boldsymbol{j}\rangle\times\langle \boldsymbol{B}\rangle$. Indeed for $\kappa\approx 1$ the averaged Lorentz force becomes conservative so that this concept would loose its relevance.

4. The turbulent transport of angular momentum (TMT)

Due to the structure of the conservation law of angular momentum,

$$\partial(\rho r^2\sin^2\theta\Omega)/\partial t+\mathrm{div}\{\rho r\sin\theta(r\sin\theta\Omega\langle \boldsymbol{u}\rangle+\boldsymbol{t}-v_\varphi\boldsymbol{v})\}=0, \quad (33)$$

with $t_i=\boldsymbol{e}_j\mathbf{Q}_{ij}^{\rm tot}$ ($\boldsymbol{e}$ eastward unit vector), the TMT is closely related to the off-diagonal correlations with exactly one φ-component. Such terms only exist in the odd part of the correlation tensor, (17), which will thus be discussed here in more detail.

Obviously, isotropic original turbulences – whose spectral tensor contains only terms such as δ_{ij} as well as k_ik_j – do not contribute to $Q^{\rm odd}$. *Neither the influence of* $\boldsymbol{B}$ *nor solid-body rotation makes isotropic turbulence relevant for TMT-effects.* It is thus *anisotropic* turbulence which is interesting in our context. As known, the most simple spectral tensor for anisotropic but homogeneous turbulence is

$$\hat{Q}_{ij}^{(0)}=q(k,\omega)\{k^2\delta_{ij}-k_ik_j-(\boldsymbol{g}\boldsymbol{k})^2\delta_{ij}-k^2g_ig_j+(\boldsymbol{g}\boldsymbol{k})(g_ik_j+g_jk_i)\}, \quad (34)$$

describing a turbulent flow with components only perpendicular to $\boldsymbol{g}$. For stellar applications the preferred direction $\boldsymbol{g}$ will naturally denote the radial one, $\boldsymbol{g}=\boldsymbol{r}/r$. Insertion of (34) into (17) yields for the components of main interest

$$\hat{\mathbf{Q}}_{r\varphi}=(\omega^2+\nu^2k^4)(\omega^2+\eta^2k^4)\{\nu(\omega^2+\eta^2k^4)+\eta(\boldsymbol{k}\boldsymbol{v})^2\}[\boldsymbol{g},\boldsymbol{e},\boldsymbol{k}]q(k^2-(\boldsymbol{g}\boldsymbol{k})^2)(2\boldsymbol{k}\boldsymbol{\Omega})/NN^*,$$

$$\hat{\mathbf{Q}}_{\theta\varphi}=(\omega^2+\nu^2k^4)(\omega^2+\eta^2k^4)\{\nu(\omega^2+\eta^2k^4)+\eta(\boldsymbol{k}\boldsymbol{v})^2\}(\boldsymbol{g}\boldsymbol{k})q(k^2-(\boldsymbol{g}\boldsymbol{k})^2-2(\boldsymbol{e}\boldsymbol{k})^2)(2\boldsymbol{k}\boldsymbol{\Omega})/NN^*. \quad (35)$$

4.1. Isotropic spectral function

If the magnetic field is weak and the spectral function only depends on the wave-number k, but not on the *vector* $\boldsymbol{k}$, transition from (35) to velocity correlations is possible and yields

$$Q_{r\varphi}=\Omega\sin\theta\int\frac{\nu k^4 q(k,\omega)}{\omega^2+\nu^2k^4}[I_1-I_2\sin^2\theta]\mathrm{d}\boldsymbol{k}\mathrm{d}\omega$$
$$+\Omega v_\varphi^2\sin\theta\int\frac{\eta k^6 q(k,\omega)}{(\omega^2+\nu^2k^4)(\omega^2+\eta^2k^4)}$$
$$\times[I_3-I_4\sin^2\theta]\mathrm{d}\boldsymbol{k}\mathrm{d}\omega,$$
$$\mathbf{Q}_{\theta\varphi}=-\Omega\cos\theta\sin^2\theta\int\frac{\nu k^4 q(k,\omega)}{\omega^2+\nu^2k^4}I_2\mathrm{d}\boldsymbol{k}\mathrm{d}\omega$$
$$-\Omega v_\varphi^2\cos\theta\int\frac{\nu k^6 q(k,w)}{(\omega^2+\nu^2k^4)(\omega^2+\eta^2k^4)}$$
$$\times\left[\frac{1}{3}I_3+I_4\sin^2\theta\right]\mathrm{d}\boldsymbol{k}\mathrm{d}\omega, \quad (36)$$

where only the dominating zonal component of the magnetic field is retained and terms of the fourth and higher orders in $\boldsymbol{v}$ are neglected. The functions $I_n(k,\omega,\Omega)$ are very complicated and we present them below only in their limiting cases. Note that these equations are valid for arbitrary rotational velocities. Similarly, using $(14)_2$ and (35), we find the correlation tensor for magnetic fluctuations;

$$\mathbf{Q}_{r\varphi}^{\mathrm{mag}}=\Omega v_\varphi^2\sin\theta\int\frac{\nu k^6 q(k,\omega)}{(\omega^2+\nu^2k^4)(\omega^2+\eta^2k^4)}$$
$$\times[I_5-I_6\sin^2\theta]\mathrm{d}\boldsymbol{k}\mathrm{d}\omega,$$
$$\mathbf{Q}_{\theta\varphi}^{\mathrm{mag}}=-\Omega v_\varphi^2\cos\theta\int\frac{\nu k^6 q(k,\omega)}{(\omega^2+\nu^2k^4)(\omega^2+\eta^2k^4)}$$
$$\times\left[\frac{1}{3}I_5+I_6\sin^2\theta\right]\mathrm{d}\boldsymbol{k}\mathrm{d}\omega. \quad (37)$$

Let us use also here the following representation of the zonal momentum fluxes (cf. Rüdiger, 1989):

$$\mathbf{Q}_{r\varphi}^{\mathrm{tot}}=\nu_T\Omega\sin\theta(V^{\mathrm{tot}(0)}+V^{\mathrm{tot}(1)}\sin^2\theta),$$
$$\mathbf{Q}_{\theta\varphi}^{\mathrm{tot}}=\nu_T\Omega\cos\theta(H^{\mathrm{tot}(0)}+H^{\mathrm{tot}(1)}\sin^2\theta). \quad (38)$$

Comparison of (36) and (37) with (38) shows that $V^{\mathrm{tot}(1)}=H^{\mathrm{tot}(1)}$ holds for arbitrary rotational values. The coincidence of $V^{\mathrm{tot}(1)}$ and $H^{\mathrm{tot}(1)}$ takes place for the complete stress tensor as well as for the Reynolds and Maxwellian stresses separately. In addition, we find for the magnetically induced terms (proportionate to v_φ^2) that $V^{\mathrm{tot}(0)}=-3H^{\mathrm{tot}(0)}$. These findings are restricted, however, to the case of a weak magnetic field and also require the assumption of horizontal isotropy of the original turbulence.

The explicit expressions for the functions I_n are rather cumbersome, and we proceed with discussion of the two extreme cases of slow and rapid rotation. The correlation tensors for these cases follow after substitution into the general results (36) and (37) of asymptotic expressions for the functions I_n. For example for $\Omega=0$ we have

$$I_1=8/15,\quad I_5=4/35,$$
$$I_3=4/35+\frac{16}{35}\left(\frac{\nu}{\eta}-\frac{\omega^2}{\omega^2+\nu^2k^4}-\frac{\nu^2k^4}{\omega^2+\nu^2k^4}\right),$$
$$I_2=I_4=I_6=0. \quad (39)$$

In the opposite case of very rapid rotation ($\Omega^2\gg\omega^2+\nu^2k^4$) it is

$$I_1=\frac{\pi}{8\Omega^3\nu k^2}(\omega^2+\nu^2k^4)^2,\quad I_2=3I_1/4,\quad I_5=3I_1/8,$$
$$I_6=3I_1/16,\quad I_3=I_4=0.$$

All terms of higher order in the parameter $(\omega^2+\nu^2k^4)/(4\Omega^2)$ are neglected here.

4.2. Slow rotation

The given results shall next be discussed for slow rotation, i.e. in a linear approximation in Ω. The magnetic field shall also be weak allowing for a series expansion in the sense of (19). These restrictions adopted we arrive at the radial TMT-expression

$$\hat{\mathbf{Q}}_{r\varphi}=[\boldsymbol{g}(\boldsymbol{e}\times\boldsymbol{k})]q(k^2-(\boldsymbol{gk})^2)(2\boldsymbol{k}\Omega)$$
$$\times\left(\frac{\nu}{\omega^2+\nu^2k^4}+\frac{\omega^2(4\nu+\eta)-3\eta\nu^2k^4)}{(\omega^2+\eta^2k^4)(\omega^2+\nu^2k^4)^2}(\boldsymbol{kv})^2\right). \quad (41)$$

If indeed the spectral function only depends on the wave-number k and the frequency ω – and not, e.g., on $(\boldsymbol{gk})$ – our above results (36) and (39) lead to

$$\mathbf{Q}_{r\varphi}/\Omega=\frac{8}{15}\int\frac{\nu k^4 q}{\omega^2+\nu^2k^4}\mathrm{d}\boldsymbol{k}\mathrm{d}\omega\sin\theta$$
$$+\frac{12v_\varphi^2}{105}\int\frac{\omega^2(4\nu+\eta)-3\eta\nu^2k^4}{(\omega^2+\eta^2k^4)(\omega^2+\nu^2k^4)^2}qk^6\mathrm{d}\boldsymbol{k}\mathrm{d}\omega\sin\theta. \quad (42)$$

The inclusion of the Maxwell stress, (37), modifies only the second integral, so that

$$\mathbf{Q}_{r\varphi}^{\mathrm{tot}}=\ldots-\frac{4\Omega}{35}v_\varphi^2\int T(k)k^6\mathrm{d}\boldsymbol{k}\sin\theta, \quad (43)$$

results with

$$T=\int\frac{\nu^2k^4(3\eta+\nu)-\omega^2(3\nu+\eta)}{(\omega^2+\eta^2k^4)(\omega^2+\nu^2k^4)^2}q\mathrm{d}\omega, \quad (44)$$

where only the dominating azimuthal magnetic field is considered. The given expression represents the magnetic feedback of the dynamo-generated field on the Λ-effect as the generator of differential rotation. We have assumed in the following discussion the spectral function as positive ("horizontal-type turbulence"), so that also $\mathbf{Q}_{r\varphi}$ (for vanishing magnetic field) is positive. As can be simply justified the opposite case of negative q is automatically involved into the considerations.

We are mainly interested on the sign of the magnetically-induced expression (44), which is *not* positive-definite but "almost always" positive. We demonstrate this finding by means of characteristic examples. The true frequency spectrum of turbulence lies somehow between the profile for very short correlation time ($q\approx$const.) and that for very large correlation time, $q\approx\delta(\omega)$. In the latter case the integral in (44) is obviously positive. On the other hand, the case of "white noise" ($\partial q/\partial\omega=0$) yields a positive quantity, too:

$$T=\pi q(k)/\nu\eta k^6. \quad (45)$$

The general case, however, is hard to consider. What is expected simply from inspection is a strong dependence on the diffusivities.

For the above mentioned case $\nu \ll \eta$ we get

$$T=\eta \int \frac{3\nu^2 k^4-\omega^2}{(\omega^2+\nu^2 k^4)^2} \frac{q}{\omega^2+\eta^2 k^4} \mathrm{d}\omega, \tag{46}$$

or, after a reformulation,

$$T=\eta \int \left\{ \frac{2\nu^2 k^4 q}{(\omega^2+\nu^2 k^4)^2} - \omega \frac{\partial}{\partial\omega} \frac{q}{\omega^2+\eta^2 k^4} \right\} \frac{\mathrm{d}\omega}{\omega^2+\nu^2 k^4}, \tag{47}$$

which is *positive* for spectra with $\partial q/\partial\omega \leq 0$ ("turbulence"). For the case of "high-conductivity", on the other hand, i.e. $\nu \gg \eta$, the integral T becomes

$$T=-\int \frac{\nu\omega}{(\omega^2+\nu^2 k^4)^2} \frac{\partial}{\partial\omega} \left\{ \frac{\omega^2+\nu^2 k^4}{\omega^2+\eta^2 k^4} q \right\} \mathrm{d}\omega, \tag{48a}$$

which is also positive for spectra of turbulence-type.

Another interesting case is $\nu=\eta$ for which also the behaviour of the integral for *vanishing* diffusivities can be studied in detail. We have then

$$T=4\nu \int \frac{\nu^2 k^4-\omega^2}{(\omega^2+\nu^2 k^4)^3} q \mathrm{d}\omega, \tag{48b}$$

which we already know from the theory of differential rotation (Rüdiger, 1989, p. 261). One of its representations is

$$T=-4\nu \int \frac{\omega}{\omega^2+\nu^2 k^4} \frac{\partial}{\partial\omega} \frac{q}{\omega^2+\nu^2 k^4} \mathrm{d}\omega, \tag{48c}$$

from which its positivity becomes apparent for turbulence-type spectra. On the other hand the structure of the kernel in (48b), suggests *negative* T-values for small viscosity. Indeed, in the limit $\nu \to 0$ the kernel in (48b) becomes a negative δ-function, $-\delta(\omega)/\omega^2$, so that the spectral function q *must* vanish at the zero-frequency $\omega=0$ in order to yield finite values. Spectra running e.g. with ω^2 at small frequencies lead to negative T for positive spectrum – in contrast to the above considered monotonically decreasing spectral functions.

We find thus opposite tendencies for the magnetic influence on the Λ-effect in dependence on the applied spectral function. Turbulence-type spectra produce a *suppression* of the radial TMT while media with vanishing dissipation require vanishing $q(k, 0)$, leading to an *amplification* of this quantity.

Let us turn next to the *horizontal* covariance $\mathbf{Q}_{\theta\varphi}$ and $\mathbf{Q}_{\theta\varphi}^{\text{tot}}$. From (42) it follows directly

$$\mathbf{Q}_{\theta\varphi}=\frac{4}{105} \int \frac{3\eta\nu^2 k^4-\omega^2(4\nu+\eta)}{(\omega^2+\eta^2 k^4)(\omega^2+\nu^2 k^4)^2} q k^6 \mathrm{d}\boldsymbol{k} \mathrm{d}\omega \Omega \cos\theta v_\varphi^2, \tag{49}$$

or

$$\mathbf{Q}_{\theta\varphi}^{\text{tot}}=\frac{4}{105} \int \frac{\nu^2 k^4(\nu+3\eta)-\omega^2(\eta+3\nu)}{(\omega^2+\eta^2 k^4)(\omega^2+\nu^2 k^4)^2} q k^6 \mathrm{d}\boldsymbol{k} \mathrm{d}\omega \Omega \cos\theta v_\varphi^2, \tag{50}$$

respectively. Due to the latitude-dependence of the magnetic field the presented correlations vanish at the poles – as it must. As before, it is also the spectral function which basically determines the sign of the correlations. For positive q (i.e. horizontal turbulence intensity dominating vertical intensity) the correlations are positive only for turbulence-type spectra, but not for such allowing the inclusion of the limit $\nu \to 0$.

4.3. Rapid rotation

Substitution of the asymptotic solutions (40) into (36) yields both the velocity covariances for rapid rotation:

$$\mathbf{Q}_{r\varphi}=(\pi/8\Omega^2)\sin\theta\left(1-\frac{3}{4}\sin^2\theta\right)\int k^2(\omega^2+\nu^2 k^4) q(k,\omega) \mathrm{d}\boldsymbol{k} \mathrm{d}\omega,$$

$$\mathbf{Q}_{\theta\varphi}=-(3\pi/32\Omega^2)\cos\theta\sin^2\theta \int k^2(\omega^2+\nu^2 k^4) q(k,\omega) \mathrm{d}\boldsymbol{k} \mathrm{d}\omega. \tag{51}$$

Similarly, for the magnetic correlations we find

$$\mathbf{Q}_{r\varphi}^{\text{mag}}=\frac{3\pi v_\varphi^2}{64\Omega^2}\sin\theta\left(1-\frac{1}{2}\sin^2\theta\right)\int \frac{\omega^2+\nu^2 k^4}{\omega^2+\eta^2 k^4} k^4 q(k,\omega) \mathrm{d}\boldsymbol{k} \mathrm{d}\omega,$$

$$\mathbf{Q}_{\theta\varphi}^{\text{mag}}=-\frac{\pi v_\varphi^2}{64\Omega^2}\cos\theta\left(1+\frac{3}{2}\sin^2\theta\right)\int \frac{\omega^2+\nu^2 k^4}{\omega^2+\eta^2 k^4} k^4 q(k,\omega) \mathrm{d}\boldsymbol{k} \mathrm{d}\omega. \tag{52}$$

The latter quantity has certainly the same sign as the integrals in (51). Obviously, the magnetic contributions are reducing the stresses.

Note that the magnetic field does not appear in (51). The reason is that the magnetic corrections of the Reynolds stress are only running with Ω^{-3} in the rapid-rotation limit.

4.4. Modest rotation

The above formalism is hard to handle if the restriction to slow rotation is relaxed. But the knowledge of higher-order terms in Ω is needed in the theory of differential solar rotation as the Coriolis number $\Omega^*=2\tau\Omega$ reaches values of order unity at the bottom of the convection zone. For the present paper we adopt thus the approximation of Orszag (1970) and Vainshtein and Kichatinov (1983) which in the language of the present paper is given by

$$q=q(k)\delta(\omega)/\tau, \quad \nu k^2=1/\tau, \quad \eta k^2=1/\tau. \tag{53}$$

The first of these relations ("τ-approximation") describes a very sharply peaked correlation spectra while the second one is of a formal nature. As described in Kichatinov (1986) it may represent the influence of nonlinear terms in the Navier-Stokes equation.

Formally, however, it implies *Reynolds numbers of order unity*. That is just the value for which the second-order correlation-approximation reaches its limitations. For such a magnetic Reynolds number dramatic consequences for the Maxwellian stress tensor have already been suggested by Rüdiger et al. (1986),

This model yields great simplifications. So the denominator (18) becomes simply

$$NN^* \approx 1+4\tau^2(\boldsymbol{k}\boldsymbol{v})^2+8\tau^2(\boldsymbol{k}\boldsymbol{\Omega}/k)^2, \tag{54}$$

and hence the TMT expressions

$$\mathbf{Q}_{r\varphi}=-2\int (1+\tau^2(\boldsymbol{k}\boldsymbol{v})^2)(k_\theta^2+k_\varphi^2)k_\theta(\boldsymbol{k}\boldsymbol{\Omega})(1-4\tau^2(\boldsymbol{k}\boldsymbol{v})^2 - 8\tau^2(\boldsymbol{k}\boldsymbol{\Omega}/k)^2)k^{-2}q(k)\mathrm{d}\boldsymbol{k},$$

$$\mathbf{Q}_{\theta\varphi}=+2\int (1+\tau^2(\boldsymbol{k}\boldsymbol{v})^2)(k_\theta^2-k_\varphi^2)k_r(\boldsymbol{k}\boldsymbol{\Omega})(1-4\tau^2(\boldsymbol{k}\boldsymbol{v})^2 - 8\tau^2(\boldsymbol{k}\boldsymbol{\Omega}/k)^2)k^{-2}q(k)\mathrm{d}\boldsymbol{k}, \tag{55}$$

result from Eqs. (35). If q depends only on the scalar quantity

k – that implies a special cell structure (!) – the remaining integrations are simple to manage. As usual we write

$$\mathbf{Q}_{r\varphi} = v_T(V^{(0)} + V^{(1)}\sin^2\theta)\Omega\sin\theta,$$
$$\mathbf{Q}_{\theta\varphi} = v_T(H^{(0)} + H^{(1)}\sin^2\theta)\Omega\cos\theta, \quad (56)$$

and obtain the TMT-coefficients

$$V^{(0)} = v_T^{-1}\int\{8/15 - 12/35\tau^2 v_\varphi^2 k^2 - 16\tau^2\Omega^2(4/35 + 2/105\tau^2 v_\varphi^2 k^2)\}k^2 q(k)\mathrm{d}\boldsymbol{k},$$
$$H^{(0)} = (4/35 + 32/315\tau^2\Omega^2)v_T^{-1}\tau^2 v_\varphi^2\int k^4 q(k)\mathrm{d}\boldsymbol{k},$$
$$V^{(1)} = H^{(1)} = -32\tau^2\Omega^2 v_T^{-1}\int\{1/35 + 1/315\tau^2 v_\varphi^2 k^2\}k^2 q(k)\mathrm{d}\boldsymbol{k}, \quad (57)$$

In particular the terms $V^{(1)}$ and $H^{(1)}$ are responsible for the latitudinal profile of non-uniform rotation. The surprising result is that they are *amplified by the large-scale magnetic field*.

Transition to the *total* tensor $\mathbf{Q}^{\mathrm{tot}}$ and its components is also interesting. Up to the second order in the Alfvén velocity we find the expressions

$$V^{\mathrm{tot}(0)} = \frac{1}{v_T}\int\left\{\frac{8}{15} - \frac{16}{35}\tau^2 v_\varphi^2 k^2 - \frac{64}{35}\tau^2\Omega^2\right\}k^2 q(k)\mathrm{d}\boldsymbol{k}, \quad (58)$$

$$H^{\mathrm{tot}(0)} = \frac{16}{105}\frac{\tau^2 v_\varphi^2}{v_T}\int k^4 q(k)\mathrm{d}\boldsymbol{k}, \quad (59)$$

$$V^{\mathrm{tot}(1)} = H^{\mathrm{tot}(1)} = -\frac{32}{35}\frac{\tau^2\Omega^2}{v_T}\int k^2 q(k)\mathrm{d}\boldsymbol{k}. \quad (60)$$

Only the fundamental modes of the TMT depend on the magnetic field while $V^{\mathrm{tot}(1)}$ as well as $H^{\mathrm{tot}(1)}$ are uninfluenced. The above stated magnetic amplification here no longer exists. We should keep this results in mind for the theory of the solar torsional oscillations.

5. Discussion: micro or macro feedback?

The Eqs. (58) . . . (60) are the main result of the present work. They reflect the influence of the magnetic field on the TMT coefficients (38). These again are responsible for the maintenance of differential rotation. Let us thus finally estimate the influence of the magnetic terms on the rotation law. Only leading terms will be taken into account. As we know, an ansatz like (38) leads in the thin-shell approximation to a surface rotation law of

$$\Omega = \Omega_0\sum_l \omega^{(l)}\sin^{2l}\theta$$

with

$$\omega^{(l)} = (d/2)V^{\mathrm{tot}(l)} + H^{\mathrm{tot}(l)}/(2l), \quad (61)$$

(cf. Rüdiger, 1989, p. 134) with d as the fractional thickness of the convection zone. In order to obtain the coefficients $V^{\mathrm{tot}(1)}$ and $H^{\mathrm{tot}(1)}$ from (58) . . . (60) a model for the magnetic field is needed. To this end we select the most simple one, i.e. the equatorially antisymmetric (dipolar) mode of the very solution:

$$\langle\boldsymbol{B}\rangle = (b_r(r)\cos\theta, \quad b_\theta(r)\sin\theta, \quad b_\varphi(r)\sin\theta\cos\theta). \quad (62)$$

The Lorentz force of this large-scale field maintains a mode of differential rotation of

$$\omega_{\mathrm{mac}}^{(2)} = -\frac{Rv_r}{10\Omega_0 v_T}\frac{\mathrm{d}v_\varphi}{\mathrm{d}r}. \quad (63)$$

On the other hand, we can read from (58) . . . (60) the results

$$V^{\mathrm{tot}(1)} = -\frac{16}{35}\frac{\tau^2}{v_T}v_\varphi^2\int k^4 q(k)\mathrm{d}\boldsymbol{k},$$
$$H^{\mathrm{tot}(1)} = \frac{16}{105}\frac{\tau^2}{v_T}v_\varphi^2\int k^4 q(k)\mathrm{d}\boldsymbol{k}, \quad (64)$$

if only the terms linear in Ω are considered. We find immediately the relation

$$\frac{\omega_{\mathrm{mic}}^{(2)}}{\omega_{\mathrm{mac}}^{(2)}} \approx 2d(\tau\Omega)(\tau^2\delta\langle\boldsymbol{u}'^2\rangle/L^2)\frac{\mathrm{d}\log r}{\mathrm{d}\log v_\varphi}\frac{v_\varphi}{v_r}, \quad (65)$$

wherein the last factor dominates for $\alpha\Omega$-dynamos. The logarithmic factor describes the inward increase of the azimuthal Alfvén speed $v_\varphi \equiv b_\varphi/(\mu\rho)^{1/2}$ which, because of the simultaneous inward increase of magnetic field *and* density, should not be too strong. With $d \approx 0.3$ and $\tau\Omega \approx 1$, the ratio (65) has minimum values of 10–100. What we thus find is a *clear dominance of the micro-scale feedback* via Reynolds stress over that of the normal macro-scale Lorentz force $\langle\boldsymbol{j}\rangle \times \langle\boldsymbol{B}\rangle$.

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