

Alpha-effect and alpha-quenching

G. Rüdiger¹ and L.L. Kichatinov^{1,2}

¹ Astrophysikalisches Institut Potsdam, An der Sternwarte 16, O-1591 Potsdam, Germany

² Institute for Solar-Terrestrial Physics, P.O. Box 4026, Irkutsk, 664033, Russia

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Abstract. New determinations of the important, but poorly known α -effect are presented in the Second-Order Correlation Approximation with both gradients of density and turbulence intensity simultaneously involved. In both of the two approaches new results appear which are – if they are confirmed by numerical simulations – relevant to the design of new dynamo models. In our first approach the magnetic back-reaction is ignored but no restrictions on the rotation rate are imposed. The density stratification as well as the turbulence intensity inhomogeneity contribute with similar power to the production of the α -effect. The known equality of both contributions appears for fast rotation only. The previously suggested change of the sign of the α -effect in the solar overshoot region may indeed be real.

Most surprising are the results for the z-component of the α -tensor. It is *negative* (in the northern hemisphere) for slow rotation and vanishes for fast rotation. While the former case is important for galaxies, the latter one concerns the majority of cool main-sequence stars.

In our second approach the magnetic feedback on the turbulence is considered which leads to the known α -quenching. We give the complete quenching functions for all α -coefficients for arbitrary strength of the magnetic field. We find different quenching for different components of the α -tensor. Additionally, the quenching strongly depends on the orientation of the mean magnetic field.

Key words: turbulence – star: magnetic field – interstellar medium: magnetic field

1. Introduction

A central part of mean-field dynamo theory is information on the turbulent electromotive force (EMF), which essentially dominates the dynamo equation:

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} - \text{rot}(\bar{\mathbf{u}} \times \bar{\mathbf{B}}) = \text{rot}(\mathcal{E} - \eta \text{rot} \bar{\mathbf{B}}) \quad (1.1)$$

Send offprint requests to: L.L. Kichatinov

with

$$\mathcal{E} = \overline{\mathbf{u}' \times \mathbf{B}'} \quad (1.2)$$

Traditionally, the EMF is split into its non-diffusive and diffusive components,

$$\mathcal{E} \cong \alpha \circ \bar{\mathbf{B}} - \eta_T \text{rot} \bar{\mathbf{B}} + \dots \quad (1.3)$$

Both defining parameters, α and η_T , deserve equal efforts of treatment. In a consistent mean-field dynamo theory, both tensors must be derived from the same turbulence model. It makes no sense to apply several quenching descriptions of the α -effect if it is not clear that the corresponding turbulence model leads to a reasonable η_T -quenching. Urgently needed in dynamo theory is a demonstration that, for a reasonable turbulence model a non-linear EMF, $\mathcal{E} = \mathcal{E}(\Omega, \bar{\mathbf{B}})$, follows allowing self-excitation.

We consider Kichatinov's (1987) "quasi-isotropic" turbulence model as a first candidate for the presented program. It contains the influence of both density stratification and a gradient of the turbulence intensity. It remains to add the influence of global rotation and large-scale magnetism in order to find the α -effect. For the derivation of the turbulent diffusivity η_T a mean field inhomogeneity must be included.

The present paper concentrates upon the non-diffusive contribution, i.e. the α -effect. There is a lack of discussion in the literature as to what is a reasonable choice for modelling cosmic dynamos. In particular, it is not definitely established whether the density or the intensity stratification contributes primarily to α , what the anisotropies are (cf. Wälder et al. 1980; Rüdiger 1990), and how the α behaves for non-slow rotation (cf. Rüdiger 1978). Knowledge of the proper choice for the magnetic quenching of the α -effect is also lacking.

More definitely, only that part of the α -tensor will be derived which is an odd function of angular velocity, representing the traditional α -effect. The even part stands for the field-advection effects and was studied elsewhere (Kichatinov 1991; Kichatinov & Rüdiger 1992 (paper I)). Our treatment may be divided into two parts. First we derive the kinematic α -expression neglecting the mean magnetic fields influence on the turbulence. No restrictions are imposed on the value of the angular velocity Ω hence we are able to consider explicitly the case of medium

rotation (Rossby number of order unity) or even the case of fast rotation with very low Rossby numbers. While the first case probably holds for accretion disks, the latter one is realized for the majority of the main-sequence stars.

Second we turn to the well-known α -quenching, i.e. the consequences of the magnetic suppression and deformation of the turbulence which generally yields a numerical decrease of the value of α . For simplicity we neglect the density stratification in both the momentum and induction equation in these computations.

2. Basic equations and turbulence model

All the derivations to follow are made within the Second Order Correlation Approximation (SOCA) which remains the principal tool of the mean-field magnetohydrodynamics. In this approximation, mean-field equations are treated in their full form, including nonlinear terms in fluctuating fields. To calculate these latter terms, however, linearized equations for the fluctuations are used. We avoid detailed discussion of SOCA because it is available elsewhere (cf, e.g. Moffatt 1978; Krause & Rädler 1981). It is known from these works that SOCA strictly applies in the case of small Strouhal numbers, $S = \tau_c u' / l_c \ll 1$ (u' is rms turbulent velocity and τ_c and l_c are the correlation time and length correspondently). Though S is usually of order unity for real conditions, the SOCA predictions are most probably correct in their order of magnitude.

As described in paper I, the turbulent EMF (1.1) follows from the linearized induction equation given in Rüdiger (1990). The mean magnetic field, $\bar{\mathbf{B}}$, is assumed to be uniform and the calculations are again restricted to the first-order terms of the scale ratio l_{corr}/L with l_{corr} and L being typical spatial scales of the fluctuating and mean field, respectively.

We refer directly to the solution of the induction equation of paper I (see its Eq. (2.4)). Considering this expression to the first order in the density gradient we obtain the following representation in the Fourier picture,

$$\begin{aligned} \rho^2 \alpha_{im} = & -\epsilon_{ipe} \int \frac{ik_m}{-i\omega + \eta k^2} \\ & \langle \hat{m}_p(\mathbf{k}, \omega) \hat{m}_e(\mathbf{k}', \omega') \rangle d\mathbf{k} d\mathbf{k}' d\omega d\omega' \\ & + \epsilon_{ipe} G_f \int \frac{\partial}{\partial k_f} \left(\frac{k_m}{-i\omega + \eta k^2} \right) M_{pe}^0(\mathbf{k}, \omega) d\mathbf{k} d\omega \\ & + \epsilon_{ipm} G_f \int \frac{M_{fp}^0(\mathbf{k}, \omega)}{-i\omega + \eta k^2} d\mathbf{k} d\omega \end{aligned} \quad (2.1)$$

with the α -tensor defined as usual,

$$\mathcal{E}_i = \alpha_{im} \bar{B}_m. \quad (2.2)$$

In (2.1) $\mathbf{m} = \rho \mathbf{u}'$ is the fluctuating momentum density with \mathbf{u}' being the random velocity field, \hat{M}^0 is the homogeneous part of the spectral tensor for the momentum fluctuations, $\mathbf{G} = \nabla \log \rho$, and the hat indicates Fourier transformation as in

$$\mathbf{m}(\mathbf{r}, t) = \int \hat{\mathbf{m}}(\mathbf{k}, \omega) e^{i(\mathbf{k}\mathbf{r} - \omega t)} d\mathbf{k} d\omega. \quad (2.3)$$

To clarify the properties of the spectral tensor $\langle \hat{m}_i \hat{m}_j \rangle$ we must address the equation of motion. As indicated above, the later equation is used in its linearized form within SOCA:

$$\begin{aligned} \partial m_i / \partial t + \nabla_i (p' + (\bar{\mathbf{B}} \cdot \mathbf{B}') / \mu) - (\bar{\mathbf{B}} \cdot \nabla) B'_i / \mu \\ - \nabla_j (\rho \nu (\nabla_i u'_j + \nabla_j u'_i) + \rho \mu' \delta_{ij} \operatorname{div} \mathbf{u}') \\ + 2\epsilon_{ips} \Omega_p m_s = f'_i, \end{aligned} \quad (2.4a)$$

where \mathbf{f} is the random body force driving the turbulence. We expect that the α -effect is not sensitive to a particular source of the turbulence and prescribe the random force, \mathbf{f} , as the source instead of addressing the rather complicated problem of nonlinear thermal convection. In the Fourier representation the above equation reads

$$\begin{aligned} [-i\omega + \nu k^2 + i\nu(\mathbf{G}\mathbf{k})] \hat{\mathbf{m}} + 2(\mathbf{k}^\circ \Omega) (\mathbf{k}^\circ \times \hat{\mathbf{m}}) \\ - \frac{1}{\mu} i (\bar{\mathbf{B}}\mathbf{k}) \hat{\mathbf{B}} = \hat{\mathbf{f}}^s, \end{aligned} \quad (2.4b)$$

where the anelasticity condition, $\operatorname{div} \mathbf{m} = 0$, was used, $\hat{\mathbf{f}}^s$ is the non-potential part of the random force, $\hat{\mathbf{B}}$ is the Fourier transform of the random component \mathbf{B}' of the magnetic field, and $\mathbf{k}^\circ = \mathbf{k}/k$ is a unit vector. We assume next that the (buoyancy) force, \mathbf{f} , do not explicitly depend on angular velocity. This allows the notion of "original turbulence" to be introduced (cf. Rüdiger 1989) and a usual linear relation be found,

$$\hat{m}_i = D_{ij} \hat{m}_j^{(0)} \quad (2.5)$$

where $\mathbf{m}^{(0)}$ is the momentum density of the original turbulence which the random force \mathbf{f} would drive if rotation and magnetic field were absent:

$$\hat{\mathbf{m}}^{(0)} = \hat{\mathbf{f}}^s / [-i\omega + \nu k^2 + i\nu(\mathbf{k}\mathbf{G})]. \quad (2.6)$$

If the Lorentz force is neglected in (2.4), we arrive at the following representation for the tensor D ,

$$D_{ij} = \frac{\delta_{ij} + \frac{2(\mathbf{k}^\circ \Omega)}{-i\omega + \nu k^2 + i\nu(\mathbf{k}\mathbf{G})} \epsilon_{ijp} k_p^\circ}{1 + \frac{4(\mathbf{k}^\circ \Omega)^2}{(-i\omega + \nu k^2 + i\nu(\mathbf{k}\mathbf{G}))^2}}, \quad (2.7)$$

If, on the other hand, the influence of the mean magnetic field on the turbulence is considered, Eq. (2.5) can still be used but the tensor D changes to

$$D_{ij}^{mag} = \frac{1}{N} \left(\delta_{ij} + 2 \frac{(\mathbf{k}^\circ \Omega)}{(-i\omega + \nu k^2)N} \epsilon_{ijp} k_p^\circ \right) \quad (2.8)$$

with

$$N = 1 + \frac{(\mathbf{k}\mathbf{V})^2}{(-i\omega + \nu k^2)(-i\omega + \eta k^2)} \quad (2.9)$$

and $V = \bar{B} / \sqrt{\mu \rho}$ as the Alfvén velocity. Equation (2.9) only holds for slow rotation and for neglected density stratification.

With (2.5) the spectral tensor for the momentum density can be expressed in terms of the spectral tensor for the "original" turbulence:

$$\langle \hat{m}_i(\mathbf{z}, \omega) \hat{m}_j(\mathbf{z}', \omega') \rangle = D_{in}(\mathbf{z}, \omega) D_{jp}(\mathbf{z}', \omega') \quad (2.10)$$

$$\langle \hat{m}_n^{(0)}(\mathbf{z}, \omega) \hat{m}_p^{(0)}(\mathbf{z}', \omega') \rangle,$$

where all the effects which a magnetic field or rotation produce in the turbulence are involved in the tensors D .

As in the Paper I, we adopt the quasi-isotropic model by Kichatinov (1987) for the original turbulence as the simplest representation for spatially inhomogeneous and divergence-free (anelastic) random fields (see also Roberts & Soward 1975):

$$\langle \hat{m}_n^{(0)}(\mathbf{z}, \omega) \hat{m}_p^{(0)}(\mathbf{z}', \omega') \rangle = \frac{\hat{E}(k, \omega, \kappa)}{16\pi k^2} \delta(\omega + \omega') \quad (2.11)$$

$$[\delta_{np} - k_n k_p / k^2 + (\kappa_n k_p - \kappa_p k_n) / 2k^2],$$

where $\mathbf{k} = (\mathbf{z} - \mathbf{z}')/2$, $\kappa = \mathbf{z} + \mathbf{z}'$. Equation (2.11) is valid within the linear approximation in the scale ratio l_{corr}/L . \hat{E} is the Fourier transform of the local spectrum E , i.e.

$$E(k, \omega, \mathbf{r}) = \int \hat{E}(k, \omega, \kappa) e^{i\kappa \mathbf{r}} d\kappa, \quad (2.12)$$

and

$$\langle m^{(0)2} \rangle = \int_0^\infty \int_0^\infty E(k, \omega, \mathbf{r}) dk d\omega. \quad (2.13)$$

It has been shown in Paper I, that the local spectrum E factorizes in the first-order approximation in the scale-ratio so that

$$E(k, \omega, \mathbf{r}) = \rho^2(\mathbf{r}) q(k, \omega, \mathbf{r}) \quad (2.14)$$

holds with q being the local velocity spectrum

$$\langle u'^2(\mathbf{r}) \rangle = \int_0^\infty \int_0^\infty q(k, \omega, \mathbf{r}) dk d\omega. \quad (2.15)$$

Only this property allows the separation of the α -effects caused by the inhomogeneity of the turbulence intensity and the density stratification.

3. The α -tensor for slow and fast rotation

We proceed with the consideration of the kinematic α -effect if the back-reaction of the magnetic field on the turbulence is neglected. The non-magnetic representation (2.7) for the tensor D is used, which is valid for arbitrary rotation rate Ω .

Due to (2.14) the resulting α tensor splits into two parts which separately involve the effects of the inhomogeneities of turbulence intensity and density:

$$\alpha = \alpha^\rho + \alpha^u.$$

Both tensors have the same structure:

$$\alpha_{im}^\rho = -\delta_{im}(\Omega \mathbf{G}) \alpha_1^\rho - (\Omega_m G_i + \Omega_i G_m) \alpha_2^\rho - (\Omega_m G_i - \Omega_i G_m) \alpha_3^\rho - \frac{\Omega_i \Omega_m}{\Omega^2} (\Omega \mathbf{G}) \alpha_4^\rho, \quad (3.1)$$

$$\alpha_{im}^u = -\delta_{im}(\Omega \mathbf{U}) \alpha_1^u - (\Omega_m U_i + \Omega_i U_m) \alpha_2^u - (\Omega_m U_i - \Omega_i U_m) \alpha_3^u - \frac{\Omega_i \Omega_m}{\Omega^2} (\Omega \mathbf{U}) \alpha_4^u, \quad (3.2)$$

where \mathbf{U} is the relative gradient of the turbulence intensity, $\nabla q/2 = \mathbf{U}q$. We assume \mathbf{U} independent of wave-number and frequency and write $\mathbf{U} = \nabla \log u'$ with u' as the rms velocity, $u' = \sqrt{\langle u'^2 \rangle}$. Then:

$$\alpha_n^u = \int_0^\infty \int_0^\infty \frac{q(k, \omega, \mathbf{r}) \eta k^4}{(\omega^2 + \nu^2 k^4)(\omega^2 + \eta^2 k^4)} \left[\nu A_n^u(\Omega, k, \omega) + \frac{\eta \omega^2}{\eta^2 k^4 + \omega^2} C_n^u(\Omega, k, \omega) \right] dk d\omega$$

for $n = 1, 4$,

$$\alpha_3^u = \alpha_3^\rho = \int_0^\infty \int_0^\infty \frac{q(k, \omega, \mathbf{r}) \omega^2}{(\omega^2 + \nu^2 k^4)(\omega^2 + \eta^2 k^4)} B_3^u(\Omega, k, \omega) dk d\omega,$$

$$\alpha_2^u = \alpha_1^u - \alpha_3^u + \int_0^\infty \int_0^\infty \frac{q(k, \omega, \mathbf{r}) \nu \eta k^4}{(\nu^2 k^4 + \omega^2)(\omega^2 + \eta^2 k^4)} \delta A^u(\Omega, k, \omega) dk d\omega,$$

$$\alpha_n^\rho = \int_0^\infty \int_0^\infty \frac{q(k, \omega, \mathbf{r}) \nu \eta k^4}{(\omega^2 + \nu^2 k^4)(\omega^2 + \eta^2 k^4)} A_n^\rho(\Omega, k, \omega) dk d\omega$$

for $n = 1, 2, 4$

The dependence on the angular velocity Ω enters the relations through the kernels A , B and C which are rather complicated expressions (see Appendix). Only special realisations shall be discussed in the present paper.

The equality $\alpha_3^\rho = \alpha_3^u$ means that both inhomogeneities can be combined in a common gradient, $\nabla \log(\rho u')$, in the antisymmetric part of the α -tensor. For the axisymmetric geometry of a star this antisymmetric part corresponds to a transport of the mean magnetic field in the azimuthal direction similar to the inducing action of differential rotation. The differential rotation, however, is more powerful than the antisymmetric part of the α -effect, so that the latter is most probably of minor importance.

The combination of the inhomogeneities of the turbulence intensity and the density into a common gradient, $\nabla \log(\rho u')$, is believed to appear for the entire α -effect (Steenbeck et al. 1966; Krause 1967). In general, however, this is not exactly true and the relative contribution of the two basic inhomogeneities depends on the angular velocity. We introduce a weight factor, S , which characterizes the relative contribution of the density inhomogeneity into the α -coefficient, $\alpha = \alpha_\varphi \varphi$, of the $\alpha\Omega$ -dynamo:

$$\alpha = -\alpha_1^u \Omega \nabla \log(\rho^S u'). \quad (3.4)$$

S depends on the angular velocity (note, however, that the ∇ -operator in (3.4) does not differentiate S).

3.1. Slow rotation

Explicit representations of the α_n^u for the slow-rotation case are given in Rüdiger (1978) and we do not reproduce them here. We concentrate upon the contribution of the density stratification:

$$\begin{aligned}\alpha_1^\rho &= \frac{4}{15} \int_0^\infty \int_0^\infty \frac{\nu \eta k^4 (3\nu^2 k^4 + 5\omega^2) q}{(\omega^2 + \nu^2 k^4)^2 (\omega^2 + \eta^2 k^4)} dk d\omega, \\ \alpha_2^\rho &= -\frac{8}{15} \int_0^\infty \int_0^\infty \frac{\nu^3 \eta k^8 q}{(\omega^2 + \nu^2 k^4)^2 (\omega^2 + \eta^2 k^4)} dk d\omega, \\ \alpha_4^\rho &= 0.\end{aligned}\quad (3.5)$$

Note the opposite signs of α_1^ρ and α_2^ρ to which we shall return below.

As we know, the α -effect also exists in perfect conductors. We speak of the high-conductivity limit if the magnetic Reynolds number of the fluctuations is much larger than unity. In this case the relations provide in the limit $\eta \rightarrow 0$

$$\begin{aligned}\alpha_1^u &= \frac{4\pi}{15\nu} \int_0^\infty q(k, 0, \mathbf{r}) k^{-2} dk, \\ \alpha_2^u &= -\frac{2\pi}{5\nu} \int_0^\infty q(k, 0, \mathbf{r}) k^{-2} dk \\ &\quad - \frac{2}{3} \int_0^\infty \int_0^\infty \frac{q(k, \omega, \mathbf{r})}{\omega^2 + \nu^2 k^4} dk d\omega, \\ \alpha_1^\rho &= \frac{2\pi}{5\nu} \int_0^\infty q(k, 0, \mathbf{r}) k^{-2} dk, \\ \alpha_2^\rho &= -\frac{4\pi}{15\nu} \int_0^\infty q(k, 0, \mathbf{r}) k^{-2} dk.\end{aligned}\quad (3.6)$$

From these relations the value

$$S = 3/2 \quad (3.7)$$

can be found for the weight factor. For the relative contribution of density and turbulence intensity to α_2 we find only an inequality, i.e.

$$\alpha_2^\rho / \alpha_2^u \leq 2/3.$$

Though the above numbers differ from unity, the difference is not large. Therefore, the importance of the two inhomogeneities for the α -effect depends mainly upon the gradients themselves.

3.2. Fast rotation

If the parameter $W = 2\Omega / \sqrt{\nu^2 k^4 + \omega^2}$ is large for those wave-numbers and frequencies which produce the dominant contributions to the integrals, we may keep only the lowest order terms in W^{-1} in (3.3). For this case of rapid rotation, $\alpha_n^u = \alpha_n^\rho = \alpha_n$, $\alpha_1 = -\alpha_4$, $\alpha_2 = \alpha_3 = 0$. Then

$$\alpha_{im} = -\alpha_1 ((\mathbf{G} + \mathbf{U})\Omega / \Omega) (\delta_{im} - \frac{\Omega_i \Omega_m}{\Omega^2}) \quad (3.8)$$

with

$$\alpha_1 = \frac{\pi\eta}{4\nu^2} \int_0^\infty \int_0^\infty \frac{(\omega^2 + \nu^2 k^4) q}{k^2 (\eta^2 k^4 + \omega^2)} dk d\omega. \quad (3.9)$$

The fast-rotation approximation is probably applicable to the deep regions of the solar convection zone. Both basic inhomogeneities combine into a common gradient in (3.8) leading to $S = I$ as the weight factor. Since the turbulence intensity inhomogeneity is dominant in the bottom layers of the convection zone – and in the overshoot region below the convection zone – one expects that the α -effect changes its sign to negative in these layers (Krivodubskij 1984).

The α -effect for the inhomogeneous intensity is known to become two-dimensional under rapid rotation (Rüdiger 1978). We notice from Eq. (3.8) that this remains valid with density stratification included. The α -effect vanishes in z -direction.

3.3. Rossby-number dependence

The above representations of the α -effect include spectral functions and other parameters which are poorly known for the solar convection zone. Hence, some simplifying assumptions are needed to derive explicit expressions. The simplest known model spectrum is

$$\begin{aligned}q(k, \omega, \mathbf{r}) &= 2 \langle u'^2 \rangle \delta(k - \ell^{-1}) \delta(\omega), \\ \nu &= \eta = \ell^2 / \tau,\end{aligned}\quad (3.10)$$

where ℓ and τ are correlation length and convective turnover time. Equation (3.10) can be understood as a transition to the mixing-length approximation (cf. Durney & Spruit 1979; Kichatinov 1991).

Inserting (3.10) into (3.3) one finds the results expressed in terms of global parameters. The non-trivial dependence is that of the Rossby number. We prefer to use the Coriolis number, $\Omega^* = 2\tau\Omega$, the inverse of the Rossby number. For the α -tensor component $\alpha_{\varphi\varphi}$, which is the basic one in $\alpha\Omega$ -dynamos, we find

$$\begin{aligned}\alpha_{\varphi\varphi} &= -\frac{1}{2} \tau \langle u'^2 \rangle \Omega^* \Psi^u(\Omega^*) \\ &\quad \frac{d}{dr} \log(\rho^{S(\Omega^*)} u') \cos \theta\end{aligned}\quad (3.11)$$

– written in spherical coordinates. The weight factor S in the above equation is a function of the Coriolis number,

$$S(\Omega^*) = \Psi^\rho(\Omega^*) / \Psi^u(\Omega^*), \quad (3.12)$$

where

$$\Psi^{\rho}(\Omega^*) = \frac{1}{\Omega^{*4}} \left(\Omega^{*2} + 6 - \frac{6 + 3\Omega^{*2} - \Omega^{*4}}{\Omega^*} \tan^{-1} \Omega^* \right),$$

$$\Psi^u(\Omega^*) = \frac{1}{\Omega^{*4}} \left(\Omega^{*2} + 9 - \frac{9 + 4\Omega^{*2} - \Omega^{*4}}{\Omega^*} \tan^{-1} \Omega^* \right).$$

In spite of the Ω^{*4} in the denominators the functions tend to constant values when Ω^* approaches zero (slow rotation):

$$\Psi^{\rho} = 4/5, \quad \Psi^u = 8/15. \quad (3.13)$$

In the opposite limit of rapid rotation ($\Omega^* \gg 1$) we have

$$\Psi^{\rho} = \Psi^u = \frac{\pi}{2\Omega^*}. \quad (3.14)$$

Note that the α -coefficient (3.11) approaches a constant value for large Ω^* , i.e. there is no quenching of the $\alpha_{\varphi\varphi}$ in the fast-rotation limit (Fig. 1).

We do not reproduce the mixing-length expression for the complete α -tensor here. It may be easily found by the use of (3.3) and (3.10). We only note that all the above findings could be qualitatively reproduced in this way. Equations (3.13) and (3.14), for instance, lead to the limits

$$\lim_{\Omega^* \rightarrow 0} S(\Omega^*) = \frac{3}{2}, \quad \lim_{\Omega^* \rightarrow \infty} S(\Omega^*) = 1, \quad (3.15)$$

for the weight factor which we have already derived above.

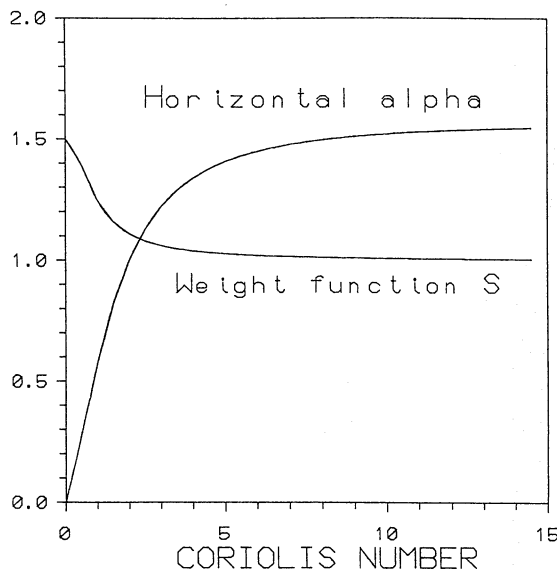


Fig. 1. The influence of the rotation rate ($\Omega^* = 2\tau\Omega$) on the $\alpha_{\varphi\varphi}$ -component which is active in $\alpha\Omega$ -dynamoes. There is no Ω -quenching. The weight-factor S approaches unity

3.4. Astrophysical applications

Let us consider the α -tensor in Cartesian geometry and for slow rotation, which is the usual constellation for galactic dynamos. In the equatorial plane the α_{xx} is active, in the vertical direction it is α_{zz} . We write

$$\alpha_{xx} = -(\alpha_x^{\rho} \mathbf{G} + \alpha_x^u \mathbf{U}) \Omega, \quad (3.16)$$

$$\alpha_{zz} = -(\alpha_z^{\rho} \mathbf{G} + \alpha_z^u \mathbf{U}) \Omega \quad (3.17)$$

and obtain in the high-conductivity limit

$$\alpha_x^{\rho} = \frac{3}{2}\hat{\alpha}, \quad \alpha_z^{\rho} = -\frac{1}{2}\hat{\alpha}, \quad (3.18)$$

$$\alpha_x^u = \hat{\alpha}, \quad \alpha_z^u = -2\hat{\alpha} - \frac{4}{3} \int_0^{\infty} \int_0^{\infty} \frac{q(k, \omega, \mathbf{r})}{\omega^2 + \nu^2 k^4} dk d\omega \quad (3.19)$$

with the positive factor

$$\hat{\alpha} = \frac{4\pi}{15\nu} \int_0^{\infty} \frac{q(k, 0, \mathbf{r})}{k^2} dk. \quad (3.20)$$

(cf. Eq. (3.6)). The occurrence of the spectral function q taken at the zero-frequency $\omega = 0$ is quite characteristic for the high-conductivity limit. It only vanishes for wave-like motions.

We find – in the northern hemisphere – the α -effect to be positive in the x -direction and *negative in the z -direction*. This is a surprising result. The galactic dynamo seems to work with different signs of α in different directions. The finding confirms the result of the numerical simulations of Brandenburg et al. (1990) who first found opposite signs for the α -effect in the two directions. Nonlinear simulations and the analytical second-order correlation-approximation lead now to the same result. Brandenburg et al. find $\alpha_x/\alpha_z \approx -0.25$. This result is reproduced here by Eq. (3.19) rather than Eq. (3.18).

Ferriere (1992) with quite another method also gives the same signs. The negativity of the α -effect in z -direction is, however, in contrast to the result in Rüdiger (1990), where the magnetohydrodynamical consequences of the action of an isotropic random force field has been analysed. Obviously, the differences between the two turbulence models are stronger than imagined.

Interesting for stellar dynamoes is also the knowledge of the α -components in spherical coordinates. From Eqs. (3.1), (3.2) and (3.6) we derive the expressions

$$\begin{aligned} \alpha_{\varphi\varphi} = \alpha_{\theta\theta} &= -\hat{\alpha} \Omega \nabla \log(\rho^{S_r} u'), \\ \alpha_{rr} &= \hat{\alpha}_r \Omega \nabla \log(\rho^{S_r} u'), \end{aligned} \quad (3.21)$$

– written in the form of (3.4). Note the different signs of the components. One easily finds that $\hat{\alpha}_r > 2\hat{\alpha}$ and $S_r < 1/4$ hence the density stratification only plays a minor role for the radial α -effect. The consequences of this unexpected behaviour are still unknown.

4. Alpha-quenching

On using Eq. (2.8), the α -tensor for slow rotation but magnetic field of arbitrary strength can be derived. It is not convenient, however, to work with tensorial representation for the nonlinear α -effect. This is because in the nonlinear case different tensor structures may provide the same contributions to the mean EMF. For this reason, we prefer here to address the mean EMF itself instead of the α -tensor.

The general nonlinear expressions are extremely bulky and we do not reproduce them here. Some simplifying cases will be considered demonstrating the main physical findings. In particular, the results for distributed turbulence intensity are developed as they seem to be of higher relevance to the galactic dynamo than the influence of density stratification (cf. Ferriere 1992).

For the weak-field limit our results approach those discussed above. It remains to consider the opposite case of strong fields.

4.1. Strong field

For the strong-field limit only those terms in the EMF must be found which do not diverge for $\bar{B} \rightarrow \infty$. No positive power in the mean magnetic field exists. Only two terms survive in this case. They do not depend on the magnetic amplitude at all:

$$\mathcal{E} = A \left((\bar{\mathbf{B}}\mathbf{U})\Omega - \frac{(\bar{\mathbf{B}}\Omega)(\bar{\mathbf{B}}\mathbf{U})}{\bar{B}^2} \bar{\mathbf{B}} \right) \quad (4.1)$$

The general expression for A is rather involved. For simplicity, the magnetic Prandtl number is assumed to be unity, $\nu = \eta$. The result is

$$A = \frac{\pi}{8\nu V} \int_0^\infty \int_0^\infty q(k, \omega, \mathbf{r}) k^{-3} dk d\omega, \quad (4.2)$$

where again $V = \bar{B}/\sqrt{\mu\rho}$ is the Alfvén velocity.

The EMF (4.1) is normal to the mean field $\bar{\mathbf{B}}$ and thus involves no effective α . It can be reduced to the advection-type term, $\mathcal{E} = \bar{\mathbf{u}} \times \bar{\mathbf{B}}$ and describes a transport of the mean field with the effective velocity

$$\bar{\mathbf{u}} = A \frac{(\bar{\mathbf{B}}\mathbf{U})}{\bar{B}^2} (\bar{\mathbf{B}} \times \Omega). \quad (4.3)$$

This EMF is not capable of producing a dynamo. For the usually assumed symmetry about the rotation axis it is strictly impossible due to the anti-dynamo theorem because of the axial symmetry of the 'velocity' (4.3).

The EMF (4.1) vanishes for parallel orientation of the angular velocity and the mean magnetic field. Remember that it also does so under fast rotation but weak field with $\bar{\mathbf{B}} \parallel \Omega$. A possible explanation of these two findings is the tendency of the turbulence field to become two-dimensional under the influence of a strong magnetic field as well as under the influence of a fast rotation. In both cases the random velocity field does not vary along the direction of the mean-field vectors. 2D flows are not influenced by the basic rotation so that no mean EMF is induced.

To find the real effect in the strong-field limit we thus only have to concentrate upon the negative powers of the magnetic field strength. This is done in the next Section in the framework of a 'mixing-length approximation'.

4.2. Mixing-length approximation

With the velocity field (3.10) we find

$$\begin{aligned} \mathcal{E} = & -\tau^2 \langle u'^2 \rangle \\ & \left(\frac{8}{15} \Psi(\beta)(\Omega\mathbf{U})\bar{\mathbf{B}} + \Psi_1(\beta) \frac{(\bar{\mathbf{B}}\Omega)(\bar{\mathbf{B}}\mathbf{U})}{\bar{B}^2} \bar{\mathbf{B}} \right. \\ & \left. - \frac{4}{5} \Psi_2(\beta)(\bar{\mathbf{B}}\Omega)\mathbf{U} - \frac{4}{5} \Psi_3(\beta)(\bar{\mathbf{B}}\mathbf{U})\Omega \right), \end{aligned} \quad (4.4)$$

with $\beta = \bar{B}/B_{eq}$ and $B_{eq} = \sqrt{\mu\rho\ell}/\tau$ as the equipartition field. The functions Ψ , Ψ_2 and Ψ_3 are normalized to unity at $\beta = 0$. They must be understood as the quenching functions for the corresponding kinematic α -coefficients.

The function Ψ ,

$$\Psi(\beta) = \frac{15}{32\beta^4} \left(1 - \frac{4\beta^2}{3(1+\beta^2)^2} - \frac{1-\beta^2}{\beta} \tan^{-1} \beta \right), \quad (4.5)$$

is of major importance because it describes the magnetic quenching of the $\alpha_{\varphi\varphi}$ -coefficient which is essential for the $\alpha\Omega$ -dynamo:

$$\alpha_{\varphi\varphi} = \alpha_0 \Psi(\beta). \quad (4.6)$$

α_0 here stands for the 'non-magnetic' value, i.e. $\beta = 0$. Note that for large β the function Ψ is proportional to β^{-3} ,

$$\Psi = \frac{15\pi}{64\beta^3}, \quad (4.7)$$

which is a stronger quenching than that described by the function $\Psi(\beta) = (1+\beta^2)^{-1}$, which is often adopted in sophisticated dynamo models. The \bar{B}^{-3} -quenching has already been established by Moffatt (1972) and by Rüdiger (1974) and has recently been reproduced by Gilbert & Sulem (1990). Also the complicated model analysis by Brestensky & Rädler (1989) leads to very similar results.

The other quenching functions in (4.4) are

$$\begin{aligned} \Psi_1(\beta) = & \frac{1}{4\beta^4} \left(\beta^2 - 5 + \frac{2\beta^2}{3(1+\beta^2)} + \frac{4\beta^4(3\beta^2-1)}{3(1+\beta^2)^3} \right. \\ & \left. + \frac{5+\beta^4}{\beta} \tan^{-1} \beta \right), \\ \Psi_2(\beta) = & \frac{5}{16\beta^4} \left(-1 + \frac{2\beta^2(3\beta^2-1)}{3(1+\beta^2)^2} \right. \\ & \left. + \frac{1+\beta^2}{\beta} \tan^{-1} \beta \right), \\ \Psi_3(\beta) = & \frac{5}{16\beta^4} \left(\beta^2 - 1 - \frac{2\beta^2(1-\beta^2)}{3(1+\beta^2)^2} \right. \\ & \left. + \frac{1+\beta^4}{\beta} \tan^{-1} \beta \right). \end{aligned} \quad (4.8)$$

In the weak-field case we find

$$\begin{aligned}\Psi &= 1 - \frac{12}{7}\beta^2, & \Psi_1 &= \frac{40}{21}\beta^2, \\ \Psi_2 &= 1 - \frac{13}{7}\beta^2, & \Psi_3 &= 1 - \frac{25}{21}\beta^2.\end{aligned}\quad (4.9)$$

All the functions (except Ψ_1) decrease with β^2 in the weak-field case. For the opposite limit of strong fields ($\beta \gg 1$) only Ψ_1 and Ψ_3 survive,

$$\Psi_1 = \pi/8\beta, \quad \Psi_3 = 5\pi/32\beta, \quad \Psi_2 = O(\beta^{-3}), \quad (4.10)$$

reproducing the result (4.1).

The essence of (4.4) appears if the α -quenching is considered in components:

$$\alpha_{xx} = -\frac{4}{15}\tau^2 \frac{d\langle u'^2 \rangle}{dz} \left(\Psi + \frac{15}{8} \frac{\bar{B}_z^2}{\bar{B}^2} \Psi_1 \right) \Omega, \quad (4.11)$$

$$\alpha_{zz} = \frac{8}{15}\tau^2 \frac{d\langle u'^2 \rangle}{dz} \left(\Psi_z - \frac{15}{16} \frac{\bar{B}_z^2}{\bar{B}^2} \Psi_1 \right) \Omega. \quad (4.12)$$

Here the combination

$$\Psi_z = -\frac{1}{2}\Psi + \frac{3}{4}\Psi_2 + \frac{3}{4}\Psi_3 \quad (4.13)$$

has been introduced. Also Ψ_z approaches unity for vanishing field. For strong fields it is proportional to $1/\beta$ quite different from Ψ . With (4.11) and (4.12) we find the α -quenching very different for different components. As the second terms in these relations demonstrate, the quenching is even stronger anisotropic. Fig. 2 displays the numerical results for the case that $\tau^2 \Omega \nabla \langle u'^2 \rangle = -1$. Notice that the quenching is highly non-uniform with the following limits:

$$\alpha_{xx} = O(\beta^{-3}) \quad \text{if} \quad \bar{\mathbf{B}} \perp \Omega,$$

$$\alpha_{xx} = O(\beta^{-1}) \quad \text{if} \quad \bar{\mathbf{B}} \parallel \Omega,$$

$$\alpha_{zz} = O(\beta^{-1}) \quad \text{if} \quad \bar{\mathbf{B}} \perp \Omega,$$

$$\alpha_{zz} = O(\beta^{-5}) \quad \text{if} \quad \bar{\mathbf{B}} \parallel \Omega.$$

In the latter case the quenching is so strong that practically no α -effect exists for $\beta > 1$ (Fig. 2).

The relatively weak decrease of the functions Ψ_1 and Ψ_3 with β in the strong-field limit as compared to that of the Ψ poses the question whether the strong-field case may be described within the $\alpha\Omega$ -dynamo. The α -coefficient $\alpha_{\varphi\varphi}$ of the $\alpha\Omega$ -dynamo is more strongly quenched than the coefficients for other directions. A more close consideration of the quenching functions, however, shows that one remains in the $\alpha\Omega$ regime if the poloidal field does not exceed the equipartition field B_{eq} irrespective of how large the toroidal field is. In any case, however, we should favour $\alpha^2\Omega$ -dynamos over $\alpha\Omega$ -dynamos.

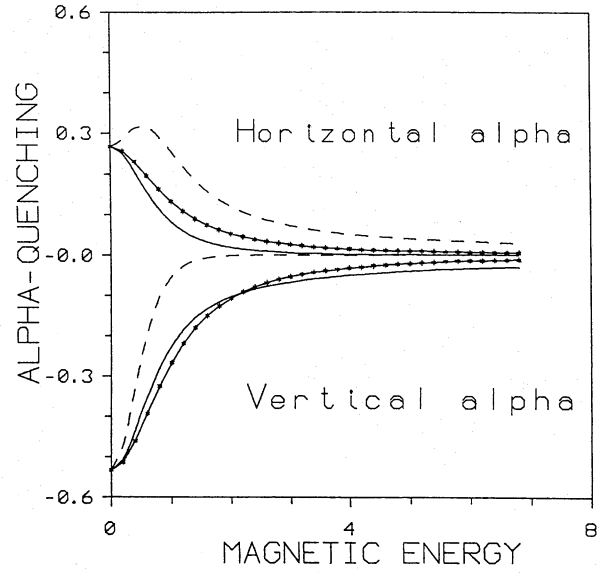


Fig. 2. The anisotropic magnetic quenching of the anisotropic alpha. Solid lines hold for $\bar{\mathbf{B}} \perp \Omega$ while dashed lines denote the case $\bar{\mathbf{B}} \parallel \Omega$. Asterisks represent the heuristic quenching function $1/(1+\bar{B}^2)$. It seems to underestimate the real quenching

Appendix

The kernels in (3.3) depend on the angular velocity, the wave-number and the frequency through two basic parameters, i.e. $W = \frac{2\Omega}{(\omega^2 + \nu^2 k^4)^{1/2}}$ and $\cos \phi = \frac{\nu^2 k^4 - \omega^2}{\omega^2 + \nu^2 k^4}$. With the abbreviations

$$LN = \log \left(\frac{W^2 - 2W \sin(\phi/2) + 1}{W^2 + 2W \sin(\phi/2) + 1} \right),$$

$$AR = \tan^{-1} \left(\frac{W - \sin(\phi/2)}{\cos(\phi/2)} \right) + \tan^{-1} \left(\frac{W + \sin(\phi/2)}{\cos(\phi/2)} \right),$$

$$\Delta = W^4 + 2W^2 \cos \phi + 1,$$

we find

$$\begin{aligned}A_1^u &= \frac{2}{W^4} \left[6 - 2 \cos \phi + \frac{W^2 + \cos \phi}{2 \cos^2(\phi/2)} \right. \\ &\quad + \frac{(1 - \cos \phi)(6 - 2 \cos \phi - W^2)}{4W \sin(\phi/2)} LN \\ &\quad + (W^4 - 4W^2 + W^2 \cos \phi (\cos \phi - 1) \\ &\quad \left. - \cos \phi (7 + 4 \cos \phi - 2 \cos^2 \phi)) \frac{AR}{4W \cos^3(\phi/2)} \right], \\ A_4^u &= \frac{2}{W^4} \left[-29 + 10 \cos \phi - \frac{3 \cos \phi}{\cos^2(\phi/2)} - \frac{(W^2 - 1)^2 (1 + W^2)}{2 \cos^2(\phi/2) \Delta} \right. \\ &\quad + \frac{(1 - \cos \phi)(-30 + 10 \cos \phi + 3W^2)}{4W \sin(\phi/2)} LN \\ &\quad + (-W^4 + 3W^2 (4 + \cos \phi - \cos^2 \phi) \\ &\quad \left. + 5 \cos \phi (7 + 4 \cos \phi - 2 \cos^2 \phi)) \frac{AR}{4W \cos^3(\phi/2)} \right], \\ \delta A &= -\frac{1}{W^2 \cos^2(\phi/2)} \left[1 + \frac{W^2 - 1}{2W \cos(\phi/2)} AR \right],\end{aligned}$$

$$\begin{aligned}
B_3^u &= \frac{1}{W^2} \left[\frac{LN}{4W \sin(\phi/2)} + \frac{AR}{2W \cos(\phi/2)} \right], \\
C_1^u &= -\frac{4}{W^4} \left[2 - \frac{W^2 - 1 + 2 \cos \phi}{4W \sin(\phi/2)} LN \right. \\
&\quad \left. - \frac{W^2 + 1 + 2 \cos \phi}{2W \cos(\phi/2)} AR \right], \\
C_4^u &= -\frac{4}{W^4} \left[-10 + \frac{3W^2 - 5 + 10 \cos \phi}{4W \sin(\phi/2)} LN \right. \\
&\quad \left. + \frac{3W^2 + 5 + 10 \cos \phi}{2W \cos(\phi/2)} AR \right], \\
A_1^\rho &= \frac{1}{2} A_1^u + \frac{1}{W^4} \left[1 + \frac{W^2 + \cos \phi}{2 \cos^2(\phi/2)} + \frac{1 - \cos \phi}{2W \sin(\phi/2)} LN \right. \\
&\quad \left. + \frac{W^4 - 2W^2 + 1 - 2 \cos \phi(1 + \cos \phi)}{4W \cos^3(\phi/2)} AR \right], \\
A_2^\rho &= A_1^\rho + \delta A, \\
A_4^\rho &= \frac{1}{2} A_4^u + \frac{1}{W^4} \left[-5 - \frac{W^2 + 5 \cos \phi}{2 \cos^2(\phi/2)} + \frac{5(\cos \phi - 1)}{2W \sin(\phi/2)} LN \right. \\
&\quad \left. + \frac{-W^4 + 6W^2 - 5 + 10 \cos \phi(1 + \cos \phi)}{4W \cos^3(\phi/2)} AR \right].
\end{aligned}$$

They are reduced in the slow-rotation limit to

$$\begin{aligned}
A_1^u &= \frac{8}{15}(2 - \cos \phi), \quad A_4^u = 0, \quad \delta A = -\frac{4}{3}, \\
B_3^u &= \frac{2}{3}, \quad C_1^u = \frac{16}{15}, \quad C_4^u = 0, \\
A_1^\rho &= \frac{4}{15}(4 - \cos \phi), \quad A_2^\rho = -\frac{4}{15}(1 + \cos \phi), \quad A_4^\rho = 0.
\end{aligned}$$

while for fast rotation

$$A_1^u = -A_4^u = -\delta A = A_1^\rho = -A_4^\rho = \frac{\pi}{2W \cos^3(\phi/2)},$$

$$B_3^u = C_1^u = C_4^u = A_2^\rho = 0.$$

holds. In the mixing-length approximation (cf. Section 3.3), $\cos \phi = 1$ and $W = \Omega^* = 2\tau\Omega$.

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