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Λ-effect and differential rotation in stellar convection zones

L.L. Kichatinov[1,2] and G. Rüdiger[1]

[1] Astrophysikalisches Institut Potsdam, An der Sternwarte 16, O-1591 Potsdam, Germany
[2] Institute for Solar-Terrestrial Physics, P.O. Box 4026, Irkutsk, 664033, Russia

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Abstract. We derive the non-diffusive part of the Reynolds stress tensor (the "Λ-effect") for the same model of inhomogeneous turbulence of rotating fluid for which the α-effect has been recently (paper I) determined. The anisotropy of the turbulence is mainly due to density stratification and the turbulence is *horizontal* in the bulk of the convection zone.

The resulting Λ-effect is very non-trivial: there are big differences between the slow and rapid rotation cases. Only a positive radial flux of angular momentum with a simple latitudinal variation exists for the former case. The corresponding angular velocity distribution shows near-independence of latitude and a distinct super-rotation in radius. For rapid rotation we find radial *and* latitudinal fluxes with more complicated profiles in latitude. The resulting rotation law is close to that provided by helioseismology. At mid-latitudes the rotation rate Ω is almost uniform trough the whole convection zone. The pole-equator difference of Ω at the bottom of convection zone is smaller than at the top. Probably almost all of the known single main-sequence cool stars belong to the "rapid rotator" class. When the angular velocity increase beyond the solar value, the relative magnitude of the differential rotation saturates at about 50 % for very small Rossby numbers.

Key words: hydrodynamics – turbulence – Sun: rotation – convection – stars: rotation

1. Motivation

Recently, we have presented several contributions in which the magnetohydrodynamic consequences of inhomogeneous turbulence under the influence of a global rotation have been studied. On this way the complete α-tensor was derived with its symmetric as well as antisymmetric parts involved. The resulting anisotropies are of fundamental importance for the dynamo theory of stellar and galactic objects.

This paper proceeds with derivations of the Reynolds stress tensor and resulting rotation laws. The primary idea is to study various effects acting in cosmic dynamos on a consistent basis. It was a usual practice to use different turbulence representations when deriving the α-effect and the Reynolds stresses though in a real star they originate from the same turbulent convection. This situation was probably caused by the absence of a turbulence model capable to produce simultaneously various effects involved in cosmic dynamos. However, the recently introduced quasi-isotropic turbulence (Kichatinov 1987) seems not to be a subject of this defect. It simultaneously accounts for the turbulence inhomogeneity and a consistent anisotropy which properties are basic for the α- and Λ-effects. The present paper is aimed to look at the quasi-isotropic turbulence "more precisely" and to derive the Λ-effect consistent with the previous derivations of α. Next, the resulting stellar rotation laws and their dependence on rotation rate are studied. We discuss whether the Λ-effect satisfies the limitations imposed by helioseismology.

Send offprint requests to: G. Rüdiger

2. The turbulence model

It is of primary importance for the mean-field hydrodynamics to define correctly the Reynolds stresses, $-<\rho u'_i u'_j>$, through which the random motions influence the mean flow. For the stellar differential rotation problem, in particular, the stresses properties brought about by global rotation and by the fluid inhomogeneity play the key role. Both effects are included in the linearized equation of motion,

$$\partial \mathbf{m}/\partial t + \nabla p' - \nabla \sigma' + 2\,\Omega \times \mathbf{m} \;=\; \mathbf{f}' \,, \tag{2.1}$$

which will be used in what follows. Here $\mathbf{m} = \rho \mathbf{u}'$ is the fluctuating momentum density, Ω is the angular velocity, $\mathbf{f}'$ is the random body force driving the turbulence, p' is the fluctuating pressure, and σ' is the viscous stress tensor,

$$\sigma'_{ij} = \rho\nu_t(u'_{j,i} + u'_{i,j}) + \rho\mu_t \,\mathrm{div}\,\mathbf{u}'\delta_{ij} \;.$$

The scalar coefficients ν_t and μ_t are due to the action of the small-scale background turbulence (cf. Stix et al. 1993). The viscosities ν_t and μ_t are assumed independent of the angular velocity. This implies that the background turbulence is vigorous enough to be insensitive to the Coriolis forces.

We assume the fluid to be inelastic,

$$\mathrm{div}\,\mathbf{m} = \;0. \tag{2.2}$$

Rewriting σ' in terms of $\mathbf{m}$ one obtains

$$\sigma'_{ij} = \nu_t \left(m_{j,i} + m_{i,j} - G_i m_j - G_j m_i\right) - \mu_t (\mathbf{G}\cdot\mathbf{m})\delta_{ij}\ , \quad (2.3)$$

with $\mathbf{G} = \nabla \log \rho$.

All our derivations belong to the quasilinear approximation in which the linearized equations for fluctuating fields are applied to derive the second-order correlations of these fields. Equation (2.1) is used to derive the Reynolds stresses. Though this approach may be a subject of criticism, it remains the basic tool of the mean-field magnetohydrodynamics. We avoid further discussion of that subject because it can be found elsewhere (cf. Moffatt 1978; Rüdiger 1989).

We assume the spatial scales of the fluctuating fields be small compared to that of the mean fields. As, however, the terms of up to second order in fluctuating to mean fields scale ratio will be included, the scale ratio must not be very small.

It is convenient by many reasons to use Fourier transforms which are introduced in accordance to

$$\mathbf{m}(\mathbf{r},t) = \int \hat{\mathbf{m}}(\mathbf{k},\omega) e^{i(\mathbf{r}\cdot\mathbf{k}-\omega t)}\, d\mathbf{k} d\omega\ .$$

Combining (2.1) and (2.3) we find

$$\hat{m}_i + \frac{2(\mathbf{k}^\circ\cdot\Omega)}{\nu_t k^2 - i\omega}\varepsilon_{ijp}k_j^\circ \hat{m}_p = \frac{\hat{f}_i^s}{\nu_t k^2 - i\omega} - \frac{i\nu_t(\mathbf{k}\cdot\mathbf{G})}{\nu_t k^2 - i\omega}\hat{m}_i -$$

$$\frac{\nu_t}{\nu_t k^2 - i\omega}\left[\left(\delta_{im} - 2k_i^\circ k_m^\circ\right) G_{mj}\hat{m}_j - G_{fp}k_f \frac{\partial}{\partial k_p}\hat{m}_i\right], \quad (2.4)$$

where the pressure has been eliminated with the condition (2.2), $\mathbf{k}^\circ = \mathbf{k}/k$ is a unit vector, and $\hat{\mathbf{f}}^s = \hat{\mathbf{f}}' - \mathbf{k}^\circ(\mathbf{k}^\circ\cdot\hat{\mathbf{f}}')$ is the solenoidal part of the force $\hat{\mathbf{f}}'$. The quantities $\mathbf{G}$ and $G_{ij} \equiv \nabla_i\nabla_j \log\rho$ must be considered as spatially uniform.

We introduce now the "original turbulence" which is defined as one that the force $\mathbf{f}'$ would produce in a non-rotating fluid. The momentum density, $\hat{\mathbf{m}}^{(0)}$, for the original turbulence obeys (2.4) with $\Omega = 0$:

$$\hat{m}_i^{(0)} = \frac{\hat{f}_i^s}{\nu_t k^2 - i\omega} - \frac{i\nu_t(\mathbf{k}\cdot\mathbf{G})}{\nu_t k^2 - i\omega}\hat{m}_i^{(0)} \quad (2.5)$$

$$-\frac{\nu_t}{\nu_t k^2 - i\omega}\left[\left(\delta_{im} - 2k_i^\circ k_m^\circ\right) G_{mj}\hat{m}_j^{(0)} - G_{fp}k_f\frac{\partial}{\partial k_p}\hat{m}_i^{(0)}\right].$$

Combining (2.4) and (2.5), we find

$$\hat{m}_i + \frac{(2\mathbf{k}^\circ\cdot\Omega)}{\nu_t k^2 - i\omega}\varepsilon_{ijp}k_j^\circ\hat{m}_p = m_i^{(0)} - \frac{i\nu_t(\mathbf{k}\cdot\mathbf{G})}{\nu_t k^2 - i\omega}\left(m_i - m_i^{(0)}\right)$$

$$-\frac{\nu_t}{\nu_t k^2 - i\omega}\Big[\left(\delta_{im} - 2k_i^\circ k_m^\circ\right) G_{mj}\left(\hat{m}_j - \hat{m}_j^{(0)}\right)$$

$$-G_{fp}k_f\frac{\partial}{\partial k_p}\left(\hat{m}_i - \hat{m}_i^{(0)}\right)\Big]. \quad (2.6)$$

This equation can be solved with a perturbation method using the above-mentioned scale ratio as a small parameter. We neglect all G-terms in (2.6) in the zeroth order in this parameter to find the known relation

$$\hat{m}_i(\mathbf{k},\omega) = D_{ij}\hat{m}_j^{(0)}(\mathbf{k},\omega) \quad (2.7a)$$

with

$$D_{ij} = \frac{\delta_{ij} + \frac{(2\mathbf{k}^\circ\cdot\Omega)}{\nu_t k^2 - i\omega}\varepsilon_{ijp}k_p^\circ}{1 + \frac{(2\mathbf{k}^\circ\cdot\Omega)^2}{(\nu_t k^2 - i\omega)^2}}. \quad (2.7b)$$

(Rüdiger 1989). Substitution of (2.7) into the RHS of (2.6) results in the solution valid to the first order in the scale-ratio:

$$\hat{m}_i(\mathbf{k},\omega) =$$

$$\left[D_{ij} - \frac{i\nu_t(\mathbf{k}\cdot\mathbf{G})}{\nu_t k^2 - i\omega}D_{ip}\left(D_{pj} - \delta_{pj}\right)\right]\hat{m}_j^{(0)}(\mathbf{k},\omega)\ . \quad (2.8)$$

The next step in the perturbation produces the expression for $\hat{m}$ valid to the second order in the scale-ratio. We shall not write out the rather complicated result.

Equation (2.8) expresses the momentum density in terms of the original turbulence. It remains to define $\hat{m}^{(0)}$. The original turbulence is assumed given. Nearly the same model for the turbulence is adopted as used before (Rüdiger & Kichatinov 1993, hereafter Paper I) to derive the α-effect. It bases on the application of the double-Fourier method (cf. Roberts & Soward 1975) to handle the large-scale inhomogeneity in space:

$$< m_i(\mathbf{r},t)m_j(\mathbf{r}+\xi,t+\tau) > = \int \hat{M}_{ij}(k,\kappa,\omega)e^{i\kappa\mathbf{r}}e^{i((\frac{\kappa}{2}+\mathbf{k})\xi-\omega\tau)}\,d\mathbf{k}\,d\kappa\,d\omega \quad (2.9a)$$

with

$$\hat{M}_{ij} = \frac{\hat{E}(k,\omega,\kappa)}{16\pi k^2}\left[\delta_{ij} - \left(1 + \frac{\kappa^2}{4k^2}\right)k_i^\circ k_j^\circ + \frac{1}{2k^2}\left(\kappa_i k_j - \kappa_j k_i\right) + \frac{\kappa_i\kappa_j}{4k^2}\right], \quad (2.9b)$$

where $\mathbf{k}$ and κ are the wave-vectors for the different scales. Now the second-order terms in the scale-ratio are included. They do not contribute to the α-effect.

The turbulence model (2.9) is practically identical to that of Paper I. Again the quantity $\hat{E}(k,\omega,\kappa)$ is the Fourier transform of the local spectrum $E(k,\omega,\mathbf{r})$:

$$E(k,\omega,\mathbf{r}) = \int e^{i\kappa\cdot\mathbf{r}}\hat{E}(k,\omega,\kappa)\,d\kappa\ ,$$

$$< m^2 > = \rho^2 < u'^2 > = \int_0^\infty E(k,\omega,\mathbf{r})\,dkd\omega\ . \quad (2.10)$$

Due to the inhomogeneities of the turbulence intensity and density the turbulence field (2.9) is anisotropic on large scales:

$$< u_x'^2 > - < u_z'^2 > = \frac{1}{8\rho^2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\int_0^\infty k^{-2}E(k,\omega,\mathbf{r})dk\,d\omega$$

$$\simeq \frac{1}{8\rho^2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}(l^2 < u'^2 > \rho^2), \quad (2.11)$$

where z and x are the vertical and a horizontal directions respectively, l is the characteristic scale of turbulent motions.

As long as $u'l \approx const$, the density profile $\rho(z)$ exclusively determines the sign of (2.11). One can easily show that in this case (2.11) is positive for the near-adiabatically stratified convection zone (Stix 1989). Hence, the turbulence proves to be *horizontal*. That, however, holds only in the bulk of the convection zone. In the top and bottom layers the behaviour of the turbulence is more complicated.

The spectral tensor (2.9) approaches the tensor for isotropic turbulence only for vanishing inhomogeneity. In this sense it was called "quasi-isotropic turbulence" by Kichatinov (1987). Equation (2.9) provides probably the simplest representation for the inhomogeneous turbulence. In Fig. 1 the 2-point correlation function is given for the horizontal velocity fluctuations u'_x. For small distances the correlation *grows* downward.

3. The Λ-effect

Rotational influence is known to enable inhomogeneous and/or anisotropic turbulence transporting angular momentum even for rigid rotation. This property named the Λ-effect is basic for stellar differential rotation (Rüdiger 1989). The angular momentum fluxes are proportional to that part of the velocity correlation tensor, $Q_{ij} =< u'_i u'_j >$, which is an odd function of the angular velocity, Ω. Application of the turbulence model (2.9) provides

$$\rho^2 Q_{ij}^{odd} =$$

$$\nu_t\,\Omega_p\left(\varepsilon_{ipf}\nabla_j + \varepsilon_{jpf}\nabla_i\right)\nabla_f \int_0^\infty \frac{EI_1}{\nu_t^2k^4+\omega^2}\,dkd\omega$$

$$-\,\nu_t\,\frac{\Omega_p\Omega_m}{\Omega^2}\left(\Omega_j\varepsilon_{ipf} + \Omega_i\varepsilon_{jpf}\right)\nabla_m\nabla_f \int_0^\infty \frac{EI_2}{\nu_t^2k^4+\omega^2}\,dkd\omega$$

$$+\,\nu_t\Omega_p\left(\varepsilon_{ipf}G_{fj} + \varepsilon_{jpf}G_{fi}\right)\int_0^\infty \frac{EI_3}{\nu_t^2k^4+\omega^2}\,dkd\omega$$

$$-\,\nu_t\,\frac{\Omega_p\Omega_m}{\Omega^2}\,G_{mf}\left(\Omega_j\varepsilon_{ipf} + \Omega_i\varepsilon_{jpf}\right)\int_0^\infty \frac{EI_4}{\nu_t^2k^4+\omega^2}\,dkd\omega$$

$$+\,\nu_t\left(\Omega_j\varepsilon_{ipf}G_p + G_j\varepsilon_{ipf}\Omega_p + \Omega_i\varepsilon_{jpf}G_p + G_i\varepsilon_{jpf}\Omega_p\right)$$

$$\nabla_f\int_0^\infty \frac{EI_5}{\nu_t^2k^4+\omega^2}\,dkd\omega$$

$$-\,\nu_t\frac{(\Omega\cdot\mathbf{G})}{\Omega^2}\Omega_p\left(\Omega_j\varepsilon_{ipf} + \Omega_i\varepsilon_{jpf}\right)\nabla_f\int_0^\infty \frac{EI_6}{\nu_t^2k^4+\omega^2}\,dkd\omega\,.$$

Generally, the kernels I_n are rather complicated nonlinear functions of the angular velocity. In the slow-rotation limit when the value of the parameter $W = 2\Omega/(\omega^2+\nu_t^2k^4)^{1/2}$ is small for those wave-numbers and frequencies which make the main contribution to the spectral integrals, they simplify considerably to read

$$I_1 = \frac{1}{30}\,\frac{\nu_t^2k^4+5\omega^2}{\nu_t^2k^4+\omega^2},\qquad I_3 = -\frac{1}{5}\,\frac{\nu_t^2k^4-\omega^2}{\nu_t^2k^4+\omega^2},$$
$$I_5 = \frac{1}{15}\,\frac{\nu_t^2k^4-\omega^2}{\nu_t^2k^4+\omega^2},\qquad I_2 = I_4 = I_6 = O(W^2). \tag{3.1}$$

In the opposite extreme of rapid rotation ($W \gg 1$) only I_2 survives while the other kernels are small:

$$I_2 = \frac{\pi}{32\Omega}\,\frac{\left(\nu_t^2k^4+\omega^2\right)^2}{\nu_t^3k^6},$$
$$I_1 = I_3 = I_4 = I_5 = I_6 = O(W^{-3})\,. \tag{3.2}$$

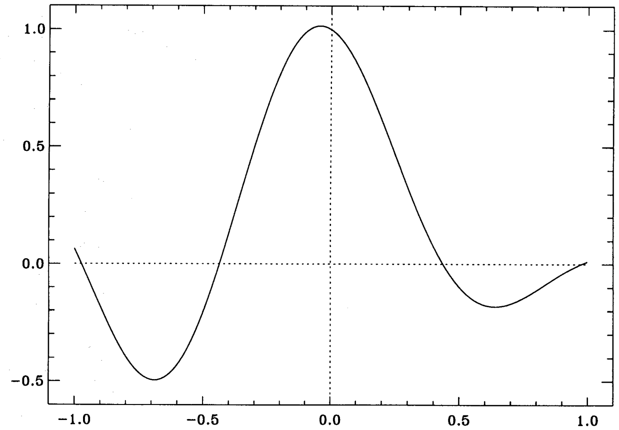

Fig. 1. A correlation function for our turbulence model (2.9). The function is normalized to unity at the origin. The two-point correlation of the horizontal fluctuations, $< u'_x(z,t)u'_x(z+\xi,t) >$, is plotted against the vertical distance ξ (normalized to the correlation length). Positive ξ mean upward displacements. The fluctuations intensity increases downward

The radial and latitudinal components of the zonal momentum flux, $\Lambda =< u'_\varphi \mathbf{u}' >$, equal to the off-diagonal components of Q_{ij}^{odd},

$$Q_{r\phi}^{odd} = \nu_t\left(V^{(0)} + \sin^2\theta V^{(1)}\right)\,\Omega\sin\theta\,,$$
$$Q_{\theta\phi}^{odd} = \nu_t H^{(1)}\,\Omega\,\sin^2\theta\,\cos\theta\,, \tag{3.3}$$

where the dimensionless functions,

$$V^{(0)} = \int_0^\infty\left[S(I_1-I_2) + S_1(I_3-I_4) + S_2(I_5-I_6)\right]\frac{dkd\omega}{\nu_t^2k^4+\omega^2},$$

$$V^{(1)} = \int_0^\infty\left[S\,I_2 + S_1I_4 + S_2I_6\right]\frac{dkd\omega}{\nu_t^2k^4+\omega^2},\quad H^{(1)} = V^{(1)}, \tag{3.4}$$

depend on the fluid inhomogeneity through the stratification characteristics,

$$S = \frac{r}{\rho^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial E}{\partial r}\right),\qquad S_1 = \frac{rE}{\rho^2}\frac{\partial}{\partial r}\left(\frac{1}{r\rho}\frac{\partial\rho}{\partial r}\right),$$
$$S_2 = \frac{1}{\rho^3}\frac{\partial\rho}{\partial r}\frac{\partial E}{\partial r}. \tag{3.5}$$

It may be noted from (3.1) and (3.4) that in the slow-rotation limit the Λ-effect redistributes angular momentum over radius only. Therefore, one may expect the radial inhomogeneity of the angular velocity being much larger than the latitudinal inhomogeneity in slowly rotating stars.

Equations (3.2) lead to the equalities $V^{(1)} = H^{(1)} = -V^{(0)}$ implying $\Lambda \parallel \Omega$ for the rapid rotators. Then disk-shaped rotation isoplanes can be expected. The rotation laws of the next section will confirm this qualitative picture.

Our expressions are still difficult to handle in models because they include spectral function which is very poorly known for stellar objects. We adopt the simplest representation for this function which can be understood as a transition to the mixing-length approximation (Kichatinov 1991):

$$E = 2\rho^2 < u'^2 > \delta(k - k_0)\, \delta(\omega)\,, \qquad \nu_t = 1/\left(k_0^2 \tau\right)\,, \tag{3.6}$$

where τ is the convective turnover time. It results in the following expressions

$$\begin{aligned} V^{(0)} = & \frac{r}{\rho^2} \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\, \tau^2 < u'^2 > \rho^2 (I_1 - I_2)\right) \\ & + \tau^2 < u'^2 > (I_3 - I_4)\, r \frac{\partial}{\partial r}\left(\frac{1}{r\rho}\frac{\partial \rho}{\partial r}\right) \\ & + \frac{1}{\rho^3}\frac{\partial \rho}{\partial r}\frac{\partial}{\partial r}\left(\tau^2 < u'^2 > \rho^2 (I_5 - I_6)\right), \end{aligned} \tag{3.7}$$

$$\begin{aligned} V^{(1)} = & \frac{r}{\rho^2} \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\, \tau^2 < u'^2 > \rho^2 I_2\right) \\ & + \tau^2 < u'^2 > I_4\, r \frac{\partial}{\partial r}\left(\frac{1}{r\rho}\frac{\partial \rho}{\partial r}\right) \\ & + \frac{1}{\rho^3}\frac{\partial \rho}{\partial r}\frac{\partial}{\partial r}\left(\tau^2 < u'^2 > \rho^2 I_6\right). \end{aligned} \tag{3.8}$$

The I_n are now functions of the Coriolis number $\Omega^* = 2\tau\Omega = 4\pi/\mathrm{Ro}$, where $\mathrm{Ro} = \tau_{rot}/\tau$ is the Rossby number. With (3.6) we find

$$\begin{aligned} I_1 &= \frac{1}{4\Omega^{*4}}\left(6 - \frac{\Omega^{*2}}{1+\Omega^{*2}} - \frac{6+\Omega^{*2}}{\Omega^*}\arctan\Omega^*\right), \\ I_2 &= \frac{1}{8\Omega^{*4}}\left(60 + \Omega^{*2} - \frac{6\Omega^{*2}}{1+\Omega^{*2}} + \frac{\Omega^{*4} - 15\Omega^{*2} - 60}{\Omega^*}\arctan\Omega^*\right), \\ I_3 &= \frac{1}{2\Omega^{*4}}\left(-3 + \frac{\Omega^{*2}}{1+\Omega^{*2}} + \frac{3}{\Omega^*}\arctan\Omega^*\right), \\ I_4 &= \frac{1}{2\Omega^{*4}}\left(-15 + \frac{2\Omega^{*2}}{1+\Omega^{*2}} + \frac{15+3\Omega^{*2}}{\Omega^*}\arctan\Omega^*\right), \\ I_5 &= \frac{1}{4\Omega^{*4}}\left(-3 + \frac{\Omega^{*2}+3}{\Omega^*}\arctan\Omega^*\right), \quad I_6 = \frac{1}{2}I_4. \end{aligned} \tag{3.9}$$

Only the odd I_n differ from zero in the slow-rotation limit ($\Omega^* \ll 1$):

$$I_1 = \frac{1}{30}, \qquad I_3 = -\frac{1}{5}, \qquad I_5 = \frac{1}{15}, \qquad I_2 = I_4 = I_6 = 0\,,$$

while for the rapid-rotation case ($\Omega^* \gg 1$)

$$I_2 = \frac{\pi}{16\Omega^*} \tag{3.11}$$

and $I_n = O\left(\Omega^{*-3}\right)$ for $n \neq 2$. The models of solar and stellar convection zones show the product $\tau < u'^2 >$ to vary slowly with depth. The Coriolis number depends on depth much less than the density does. Equations (3.7) and (3.8) can be further simplified by neglecting the contributions of all inhomogeneities except the density stratification. This yields

$$V^{(0)} = \frac{\tau^2 < u'^2 >}{H_\rho^2}\left[\mathcal{I}_0(\Omega^*) + \mathcal{I}_1(\Omega^*)\right], \tag{3.12a}$$

$$V^{(1)} = H^{(1)} = -\frac{\tau^2 < u'^2 >}{H_\rho^2}\,\mathcal{I}_1(\Omega^*)\,, \tag{3.12b}$$

where

$$H_\rho = -\frac{\mathrm{d}\, r}{\mathrm{d}\log\rho}$$

is the density scale-height. Only two functions of the Coriolis number are still involved, i.e.

$$\begin{aligned} \mathcal{I}_0 &= 4I_1 + 2I_5 = \frac{1}{2\Omega^{*4}}\left(9 - \frac{2\Omega^{*2}}{1+\Omega^{*2}} - \frac{\Omega^{*2}+9}{\Omega^*}\arctan\Omega^*\right), \\ \mathcal{I}_1 &= -4I_2 - 2I_6 = \\ &= -\frac{1}{2\Omega^{*4}}\left(45 + \Omega^{*2} - \frac{4\Omega^{*2}}{1+\Omega^{*2}} + \frac{\Omega^{*4} - 12\Omega^{*2} - 45}{\Omega^*}\arctan\Omega^*\right). \end{aligned}$$

Due to differences in the assumptions, these functions differ in details but not in their basic features from that found by Kichatinov (1987). We give finally limits, i.e.

$$\mathcal{I}_0 = 4/15 - 16\Omega^{*2}/35, \quad \mathcal{I}_1 = 16\Omega^{*2}/105 \tag{3.13a}$$

for slow rotation and

$$\mathcal{I}_0 = O(\Omega^{*-3}), \quad \mathcal{I}_1 = -\frac{\pi}{4\Omega^*} \tag{3.13b}$$

for very rapid rotation.

In Fig. 2 the main results for the Λ-effect are shown. There are drastic differences between the regions of slow and fast rotation, but almost always the product $V^{(0)}V^{(1)}$ is negative (cf. Rüdiger 1989). For non-slow rotation we find *negative* $V^{(0)}$ and *positive* $V^{(1)}$ in excellent agreement with numerical simulations by Brandenburg et al. (1990) and by Pulkkinen et al. (1990). The positive $H^{(1)}$ agree well with the observations of the horizontal random motions of the large sunspot groups (Ward 1965).

Note that the pronounced changes in the functions behavior occur at the Coriolis number values of order unity. By this reason the Ω^* is a more convenient parameter to distinguish between the slow and fast rotation regimes (cf. Durney & Spruit 1979) than the often used inverse Rossby number. E.g., the Sun with $\mathrm{Ro}^{-1} \lesssim 1$ is in effect a rapid rotator with $\Omega^* \gtrsim 1$.

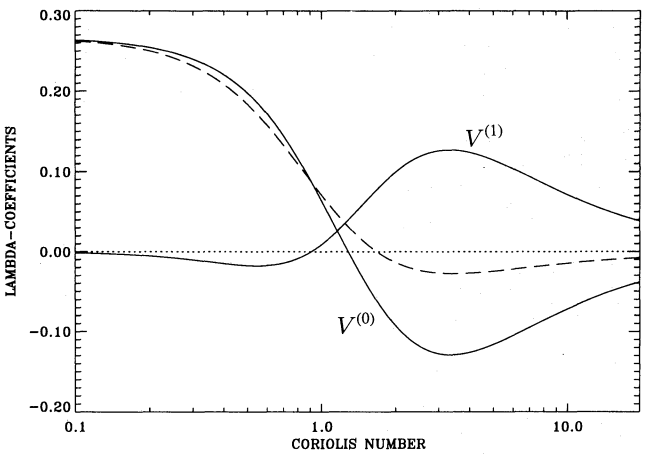

Fig. 2. $V^{(0)}$ and $V^{(1)} = H^{(1)}$ as functions of the Coriolis number $\Omega^* = 2\tau\Omega$. Note the drastic differences between slow and fast rotation. The dashed line gives the sum $V^{(0)} + \frac{4}{5}V^{(1)}$ whose meaning is given in the text

4. Rotation laws

Only the simplest form of the Reynolds equation,

$$\frac{\partial}{\partial x_i}(x \sin\theta\, \rho\, Q_{i\phi}) = 0, \tag{4.1}$$

was solved in the region $0.75 \leq x \leq 1$ with stress-free boundaries; $x = r/R$ is the fractional radius.

The total Reynolds stresses are given by

$$Q_{r\phi} = Q^{odd}_{r\phi} - \nu_\perp r \sin\theta \frac{\partial\Omega}{\partial r} - (\nu_\| - \nu_\perp)\sin\theta\cos\theta\left(r\cos\theta\frac{\partial\Omega}{\partial r} - \sin\theta\frac{\partial\Omega}{\partial\theta}\right), \tag{4.2a}$$

$$Q_{\theta\phi} = Q^{odd}_{\theta\phi} - \nu_\perp \sin\theta \frac{\partial\Omega}{\partial\theta} - (\nu_\| - \nu_\perp)\sin^2\theta\left(\sin\theta\frac{\partial\Omega}{\partial\theta} - r\cos\theta\frac{\partial\Omega}{\partial r}\right), \tag{4.2b}$$

where we take into account that for a rotating fluid different eddy viscosities apply to the directions parallel ($\nu_\|$) and normal ($\nu_\perp$) to the rotation axes. Under the approximation (3.6) the viscosities read

$$\nu_\| = \nu_T\Psi_\|(\Omega^*), \qquad \nu_\perp = \nu_T\Psi_\perp(\Omega^*), \tag{4.3}$$

where $\nu_T = 4\tau < u'^2 > /15$ is the effective viscosity for a non-rotating fluid, and the two new functions of the Coriolis number read

$$\Psi_\| = \frac{15}{32\Omega^{*4}}\left(21 - 3\Omega^{*2} + \frac{4\Omega^{*4}}{1+\Omega^{*2}} - \frac{21 + 4\Omega^{*2} - \Omega^{*4}}{\Omega^*}\arctan\Omega^*\right), \tag{4.4a}$$

$$\Psi_\perp = \frac{15}{128\Omega^{*4}}\left(\Omega^{*2} - 21 - \frac{8\Omega^{*2}}{1+\Omega^{*2}} + \frac{21 + 14\Omega^{*2} + \Omega^{*4}}{\Omega^*}\arctan\Omega^*\right) \tag{4.4b}$$

(cf. Kichatinov 1988). Equations (4.3) and (4.4) account for the rotationally induced anisotropy and quenching of the eddy viscosities. The two viscosities coincide for a non-rotating fluid, $\Psi_\perp(0) = \Psi_\|(0) = 1$, whereas $\nu_\|$ is about four times larger than $\nu_\perp$ for $\Omega^* \gg 1$. Both viscosities generally decrease with Ω^*.

In a further approximation we accept the equality, $\nu_t = \nu_T$.

A grid-point method with 21 grid points in radius and 25 in latitude (per hemisphere) was applied to solve (4.1). In all calculations the expressions (3.12) were used with the approximation

$$\tau^2 < u'^2 > \simeq l^2 \simeq H_\rho^2 . \tag{4.5}$$

This implies the value of $\alpha \simeq 1.67$ for the free coefficient of the mixing-length relation, $l = \alpha H_p$. We neglect the depth variation of H_ρ and use the value of $\mathrm{d}\log\rho/\mathrm{d}x = -8$ for the density gradient at the depth of about $x = 0.75$ in the solar convection zone for the entire computation domain. The position-dependence of the Ω^*-parameter is also neglected. We adopt these approximations in the present discussion of the stellar rotation laws predicted from the theory. The models with a detailed account for the convection zone stratification are more complicated (Küker et al. 1993).

4.1. Slow rotation

Slow rotation ($\Omega^* < 1$) leads to high positive values of $V^{(0)}$ and very small $V^{(1)} = H^{(1)}$ (Fig. 2). The resulting rotation laws possess a pronounced radial gradient but they do not exhibit any significant pole-equator differences. A very small equatorial deceleration can be observed. The outer regions rotate faster than the inner ones (~"super-rotation", cf. Fig. 3). This should be the general rotation law in old stars with slowly rotating convection zones. We find that the relative magnitude of differential rotation, $(\Omega_{top} - \Omega_{bottom})/\Omega_{top}$, saturates for very small Ω^* at the value of about 0.07.

These results might be of interest because at least some of the old evolving stars in synchronized binary systems belong to this class of slow rotators (Basri 1985). Strikingly enough, we do not know any single main-sequence star which is sufficiently old to satisfy the above condition for the Coriolis number.

4.2. Rapid rotation

Nearly all known single main-sequence stars are rapid rotators with $\Omega^* \gtrsim 1$ (Basri 1985). The $V^{(0)}$-mode becomes negative in this case and its magnitude decreases. Simultaneously the $V^{(1)}$ grows to positive values of the same order (cf. Fig. 2). This is the main condition for the maintenance of an appreciable pole-equator difference in the rotation rate (Rüdiger 1989). At the same time the sum $V^{(0)} + \frac{4}{5}V^{(1)}$ attains very small values.

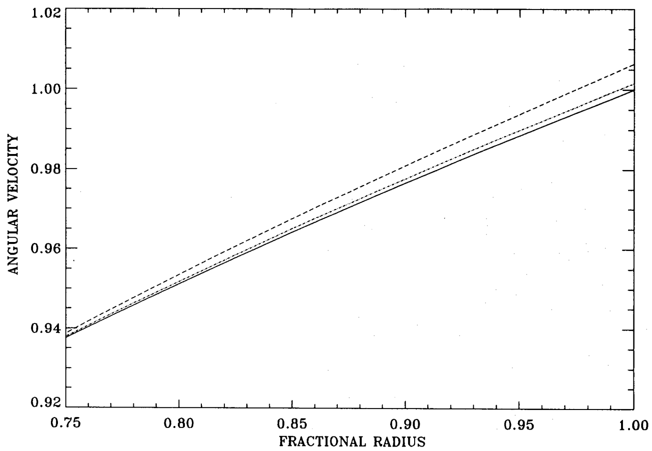

Fig. 3. The rotation law for a slow rotator, $\Omega^* = 0.5$. The solid line gives the rotation law below the equator, the dashed-dotted one - below the latitude of 30°, and the dashed line belongs to the poles. Note the weak equatorial *deceleration*

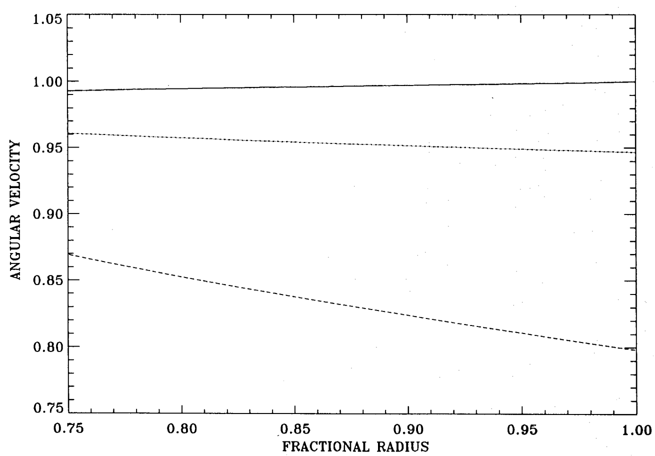

Fig. 4. The rotation law for a rapid rotator, $\Omega^* = 2$. The depth-dependences of Ω are shown for equator (solid), pole (dashed), and 30° latitude (dashed-dotted)

The radial gradient of the latitude-averaged angular velocity, $\omega_0 = (3/4) \int_0^\pi \Omega \sin^3\theta \, d\theta$, is determined by the expression

$$\frac{d\omega_0}{dr} = \left(V^{(0)} + \frac{4}{5} V^{(1)}\right) \frac{\omega_0}{r} + \text{....merid.flow.} \quad (4.6)$$

Without meridional flow and with $V^{(0)} + \frac{4}{5}V^{(1)} \approx 0$ the ω_0 remains uniform throughout the convection zone. The results of helioseismology are highly compatible with a depth-independence of the ω_0-function (Rüdiger& Tuominen 1990). No wonder that the rotation law computed with $\Omega^* = 2$ (Fig. 4) is very similar to the results of helioseismic inversions (cf. Libbrecht 1988; Dziembowski et al. 1989). Ω is nearly uniform with depth below the equator while there is strong sub-rotation below the poles. The pole-equator difference in Ω at the bottom of the convection zone is smaller than at the surface.

In Fig. 5 the run of the pole-equator difference of Ω with the inverse Rossby number Ω^* is given. The rotation at the bottom is always more rigid than that at the top. The surface differential rotation saturates at a value of about 50% for very rapid rotators. This is because of the Λ-effect (3.12) and the eddy viscosities (4.3) decrease with increasing Ω in equal proportion to the Ω^{*-1}. The absolute value of the latitudinal differential rotation, however, increases with Ω for the rapid rotators. This result may be of definite interest for simulations of magnetic activity of the rapidly rotating stars.

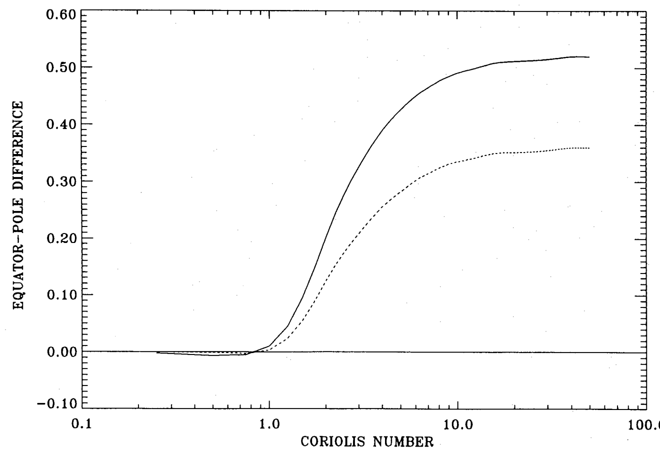

Fig. 5. The pole-equator difference in the rotation rate, $(\Omega_{eq} - \Omega_{pole})/\Omega_{eq}$, at the top (solid) and bottom (dashed) of convection zone as a function of the Ω^*

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