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A theory of transverse small-scale standing Alfvén waves in an axially symmetric magnetosphere

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Abstract. For a model of an axially symmetric magnetosphere we have constructed a theory of standing Alfvén waves with large azimuthal wavenumbers, $m \gg 1$. It is supposed that the source for such waves can be provided by extraneous currents in the ionospheric *E*-layer. A monochromatic source excites an oscillation of the poloidal type on the magnetic shell, having an eigenfrequency of poloidal oscillation that coincides with the source frequency. This wave, while travelling across the magnetic shells, changes from the poloidal into the toroidal wave. On the magnetic shell, whose frequency of toroidal eigenoscillations coincides with the frequency of the wave concerned, it is totally absorbed due to energy dissipation in the ionosphere.

1. Introduction

The purpose of this paper is to construct a theory of standing Alfvén waves which are of small scale across the magnetic field both in the azimuthal direction and normal to the magnetic shells. The transverse small-scale character is with regard to the smallness of the typical scales of variation of the wave field in the transverse directions as compared with the typical scale of variation of the magnetospheric parameters, as well as compared with the longitudinal wavelength. In particular, this means that the azimuthal wave number $m \gg 1$.

The transverse small-scale character is a natural property of the Alfvén oscillations in a transversally inhomogeneous plasma (Timofeev, 1979; Mazur *et al.*, 1979). It is due to the smallness of the transverse dispersion of the Alfvén waves (which is totally absent in an ideal MHD approximation and in a homogeneous magnetic field). The smallness of the dispersion leads to the fact that neighbouring field lines oscillate nearly independently of each other and, in the presence of a transverse inhomogeneity, with a different frequency. As a result, the oscillations scatter in phase, that is, the transverse structure of the oscillation field is reduced in size. Only the transverse dispersion can stop or prevent this comminution, but due to its smallness, this requires that the transverse wavelength should be small. However, the above reasoning proves the small-scale character of the Alfvén wave in the direction normal to the magnetic shells. In the azimuthal direction the system is assumed homogeneous, and in this direction the wave can be both a small- and large-scale one. Which of these two possibilities is realized is determined by the wave excitation mechanism.

An important example of the excitation of an azimuthally large-scale wave is Alfvén resonance, which was discovered by Southwood (1974) and Chen and Hasegawa (1974), and was studied in many subsequent papers [see a review by Southwood and Hughes (1983)], specifically for an axially symmetric magnetospheric model in the papers of Leonovich and Mazur (1989a,b). The source of the Alfvén oscillations in this case is provided by a fast magnetosonic wave which is generated outside or on the magnetospheric boundary and subsequently penetrates its depth. Such a wave has a global character, and its expansion in terms of azimuthal harmonics contains largely terms with $m \sim 1$.

Alfvén waves with $m \gg 1$ can be excited either by instabilities capable of effectively generating transversally small-scale oscillations or by local sources in the magnetosphere or the ionosphere. Such a source may be exemplified by extraneous currents produced owing to the motion of neutrals in the *E*-layer. It is likely that the combined action of these mechanisms, i.e. extraneous currents in the ionosphere gives rise to a priming disturbance which is subsequently enhanced by the instability and reaches the experimentally observed level.

A theoretical investigation of azimuthally small-scale Alfvén waves was initiated in a pioneering paper by Dungey (1954) who considered the limit $m \to \infty$. Eigenmodes, the so-called poloidal Alfvén waves, determined in this limit, are concentrated on the separated magnetic surfaces determined by the mode frequency (in this paper these surfaces

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are referred to as the poloidal resonant surfaces). A disturbed magnetic field of the poloidal mode oscillates in the direction normal to the magnetic shell, and the electric field oscillates azimuthally. In accordance with general properties of the Alfvén wave, such a polarization implies that the azimuthal component of the wavevector is much larger than the normal component. This is by no means evident from the Dungey solution, because for it both these components are infinite. The longitudinal (along the geomagnetic field line) structure and the frequency spectrum of the poloidal mode are determined by the solution of a one-dimensional problem for eigenvalues (Radosky, 1967; Radosky and Carovillano, 1969; Cummings *et al.*, 1969; Krylov *et al.*, 1981; Krylov and Lifshitz, 1984). Walker (1987) and Taylor and Walker (1987) analysed the influence upon the longitudinal structure of the poloidal mode of the finite plasma pressure.

Poloidal oscillations with large but finite m were studied by Leonovich and Mazur (1990). By considering such m values, it became possible to investigate the fine structure of the mode in the direction normal to the magnetic surfaces. It appeared that this structure is determined by the specific transverse dispersion caused by the curvature of the geomagnetic field lines. It was found that near the poloidal resonant surface, the wavelength normal to it is much larger than in azimuth, which is consistent with the poloidal polarization of the mode. But in regard to two important points, work was not completed. Firstly, the excitation mechanism for these modes was not considered. And secondly, the fate of the poloidal mode as it moves farther away from the poloidal resonance surface was not studied. The point here is that the transverse dispersion leads to the propagation of the wave across the magnetic shells, as a consequence of which the wavelength in the normal direction decreases, and the mode loses its poloidal character.

These questions have been solved in the present paper. We have developed a complete theory of the phenomenon, and its main elements may be formulated as follows. In the longitudinal direction the mode is a standing wave, and in the transverse direction it is a travelling wave. Extraneous currents in the ionosphere excite the mode near the poloidal resonance surface, and from it the wave travels toward the magnetic shell which we have called the toroidal resonance surface. In the process of such a motion, the wavelength in the normal decreases, and becomes comparable with and then less than the azimuthal one. The poloidal polarization of the mode is replaced with the toroidal polarization. At the same time, the mode amplitude decreases owing to the dissipation in the ionosphere. If there is some instability that is stronger than the ionospheric dissipation, then the mode amplitude increases. The transverse propagation of the mode ends on the toroidal resonance surface where the mode is fully absorbed (see Fig. 1).

Let us give a brief account of the organization of this paper. In Section 2, in terms of an approximation of ideal magnetic hydrodynamics of cold plasma, we derive the partial differential equation describing a transversally small-scale monochromatic Alfvén wave in an axisymmetric magnetosphere. Section 3 is devoted to the boundary condition on the ionosphere which takes into account the dissipation and the possible existence of extraneous currents there. Based on the transverse small-scale character of the oscillations concerned, in Section 4 we apply to the starting equation the WKB approximation in the coordinate normal to the magnetic surface. In this section we consider the main order of approximation that defines the quasiclassical wavevector as the solution of the longitudinal problem for eigenvalues. In Section 5 this problem is investigated qualitatively; specifically, a definition is given to the poloidal and toroidal longitudinal modes and to respective resonance surfaces. Section 6 is concerned with the solution of the longitudinal problem using different methods. Results of numerical calculations for a given magnetospheric model are described. Explicit solutions will be obtained for longitudinal harmonics with large numbers using the WKB approximation in the longitudinal coordinate. In Section 7 we examine the damping of the longitudinal modes owing to the dissipation in the ionosphere. Section 8 is devoted to the next order of WKB approximation in the transverse coordinate, in which the change in amplitude of the mode is determined as it moves across magnetic shells. Also, the validity range of the transverse WKB approximation is considered, and it is found that it is not satisfied in small neighbourhoods of the poloidal and the toroidal resonance surfaces. In the next two sections the solutions near these surfaces are obtained using the method of perturbation theory based on the smallness of deviations of the desired solutions from the poloidal and the toroidal modes, respectively. In Section 11 the solutions are matched in different regions, thereby constructing a full solution in the entire region of its existence. Some preliminary remarks on comparison of our theory with experiments are made in Section 12. In the Conclusion, the main results of this work are formulated.

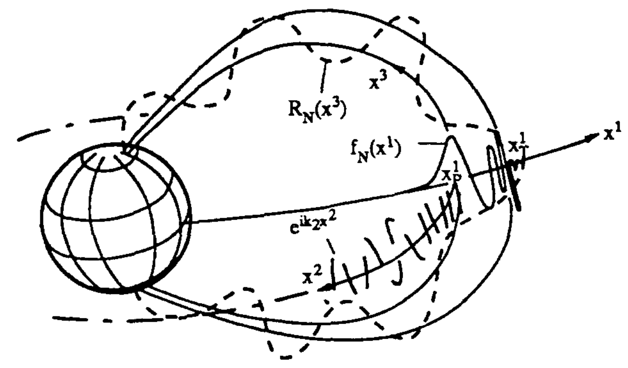

Fig. 1. Spatial structure of the transversally small-scale standing Alfvén wave in the axisymmetric magnetosphere (schematic). The functions R_N and f_N describe the structure of the mode, respectively along (coordinate x^3) and across (coordinate x^1) the field line, and in azimuthal coordinate the structure of the mode is defined by the expression $\exp(ik_2x^2)$. North–South symmetry is not assumed

2. Basic equations and relationships

For describing the axisymmetrical magnetosphere, we shall make use of an orthogonal curvilinear coordinate system x^1, x^2, x^3, in which the coordinate surfaces

$x^1 =$ const. coincide with magnetic shells, the coordinate x^2 specifies the field line on a given shell, and the coordinate x^3 specifies a point on a given field line (see Fig. 1). In the axisymmetric system, it is natural to use the azimuthal angle φ as the coordinate x_2. However, we shall, instead, be using a more general symbol x^2, assuming that x^2 may even not coincide with φ. We denote by x^3_- and x^3_+ the coordinates of intersection of the field line with the ionosphere of the magneto-conjugate hemispheres. These quantities depend on the magnetic shell: $x^3_\pm = x^3_\pm(x^1)$. The coordinate surfaces $x^3 =$ const. include a separatrix, and it is natural to call it the equatorial surface. Put $x^3 = 0$ on it and assume $x^3_- < 0$, $x^3_+ > 0$. The axial symmetry of the magnetosphere does not imply the presence of North–South symmetry, that is, the symmetry with respect to the replacement $x^3 \to -x^3$.

The length element ds in the orthogonal curvilinear coordinate system is specified by the quadratic form:

$$ds^2 = g_1\,dx^{1^2} + g_2\,dx^{2^2} + g_3\,dx^{3^2},$$

where $g_i = g_i(x^1, x^3)$ are diagonal elements of the metric tensor, and $g = g_1 g_2 g_3$ is its determinant. On each given field line it is possible to use, instead of the coordinate x^3, the physical length ℓ, whose differentials are related by the relation $d\ell = \sqrt{g_3}\,dx^3$. Put $\ell = 0$ on the equatorial plane and introduce the quantities $\ell_\pm = \ell_\pm(x^1)$ corresponding to the coordinates $x^3_\pm$. It should be borne in mind that the coordinate system (x^1, x^2, ℓ) is not an orthogonal one.

A perturbed electromagnetic field of the monochromatic oscillation with a time dependence of the form $\exp(-i\omega t)$ obeys the equations:

$$\operatorname{curl}\mathbf{E} = i\frac{\omega}{c}\mathbf{B}, \qquad \operatorname{curl}\mathbf{B} = -i\frac{\omega}{c}\hat{\varepsilon}\mathbf{E},$$

where $\hat{\varepsilon}$ is the dielectric permittivity tensor. In the approximation of ideal magnetohydrodynamics, to which we shall confine ourselves, for the cold plasma this tensor is diagonal, and its physical components (i.e. the components in a local Euclidean basis) are given by the equalities (Akhiezer *et al.*, 1974):

$$\varepsilon_{11} = \varepsilon_{22} = c^2/A^2, \qquad \varepsilon_{33} = -\infty.$$

Here, $A = A(x^1, x^3) = B_0/\sqrt{4\pi\rho}$ is the Alfvén velocity. From the above relationships it is easy to obtain a known system of equations for covariant components of electric fields E_i. Infinite longitudinal plasma conductivity ($\varepsilon_{33} = -\infty$) leads to the equality $E_3 = 0$. For the remaining two components, we have:

$$\frac{\partial}{\partial x^2}\frac{g_3}{\sqrt{g}}\frac{\partial E_1}{\partial x^2} - \frac{\partial}{\partial x^2}\frac{g_3}{\sqrt{g}}\frac{\partial E_2}{\partial x^1} + \frac{\partial}{\partial x^3}\frac{g_2}{\sqrt{g}}\frac{\partial E_1}{\partial x^3} + \frac{\sqrt{g}}{g_1}\frac{\omega^2}{A^2}E_1 = 0, \quad (1a)$$

$$-\frac{\partial}{\partial x^1}\frac{g_3}{\sqrt{g}}\frac{\partial E_1}{\partial x^2} + \frac{\partial}{\partial x^1}\frac{g_3}{\sqrt{g}}\frac{\partial E_2}{\partial x^1} + \frac{\partial}{\partial x^3}\frac{g_1}{\sqrt{g}}\frac{\partial E_2}{\partial x^3} + \frac{\sqrt{g}}{g_2}\frac{\omega^2}{A^2}E_2 = 0. \quad (1b)$$

The system of equations (1) can be represented in a convenient symmetric form. For this purpose, we introduce the following notations. To an arbitrary two-component covariant vector a_i ($i = 1, 2$), we compare the contravariant vector $\tilde{a}^i$ according to the rule $\tilde{a}^1 = a_2$, $\tilde{a}^2 = -a_1$. In particular, to a two-dimensional gradient $\nabla_i = \partial/\partial x^i$ ($i = 1, 2$), we compare the operator $\tilde{\nabla}^i = (\nabla_2, -\nabla_1)$. The identity $a_i\tilde{a}^i \equiv 0$, in particular $\nabla_i\tilde{\nabla}^i \equiv 0$ holds. Besides, we introduce a covariant vector $\tilde{a}_i = g_{ij}\tilde{a}^j$, where g_{ij} is a metric tensor. In other words, $\tilde{a}_1 = g_1 a_2$, $\tilde{a}_2 = -g_2 a_1$. We have the identity $\tilde{a}_i\tilde{a}^i = g_\perp a_\perp^2$, where $g_\perp = g_1 g_2$ and $a_\perp^2 = a_1^2/g_1 + a_2^2/g_2$ is the square of the physical length of a two-component vector $\mathbf{a}$. Using these notations we represent the system (1) as:

$$(\hat{P}^{ij} + \hat{L}^{ij})E_j = 0. \quad (2)$$

Here:

$$\hat{P}^{ij} = \tilde{\nabla}^i \frac{g_3}{\sqrt{g}} \tilde{\nabla}^j; \qquad \hat{L}^{ij} = \delta^{ij}\hat{L}^{(i)};$$

$$\hat{L}^{(i)} = \frac{\partial}{\partial x^3}\frac{1}{g_3}\frac{\sqrt{g}}{g_i}\frac{\partial}{\partial x^3} + \frac{\sqrt{g}}{g_i}\frac{\omega^2}{A^2}$$

and δ^{ij} is a unit tensor (Kronecker's symbol).

The system of equations (2) for the transversally small-scale oscillations of our interest can be reduced to a single equation. To do this, we invoke perturbation theory based on the fact that the transverse scale of the oscillation is much smaller than the longitudinal scale and, therefore, the operator $\hat{L}^{ij}$ is small compared with the operator $\hat{P}^{ij}$. By expanding the desired solution:

$$E_i = E_i^{(0)} + E_i^{(1)} + \ldots,$$

as a series of perturbation theory, in the main order from equation (2) we have:

$$\hat{P}^{ij}E_j^{(0)} = 0.$$

A general solution of this equation is:

$$E_i^{(0)} = -\nabla_i\Phi,$$

where Φ is an arbitrary function of coordinates. The function Φ is a scalar perturbation potential.

The dependence of the potential Φ on coordinates is not defined in the main order of perturbation theory. The equation for it can be obtained as a solvability condition for the equation for a next approximation correction. We linearize equation (2):

$$\hat{P}^{ij}E_j^{(1)} - \hat{L}^{ij}\nabla_j\Phi = 0. \quad (3)$$

After acting on this relationship from the left by the operator ∇_i, we obtain the desired equation for Φ:

$$\nabla_i\hat{L}^{ij}\nabla_j\Phi = 0.$$

A form that is more convenient for further use can be imparted to the equation obtained, if instead of the variable x^3, we use the variable ℓ:

$$(\nabla_1\hat{L}_T\nabla_1 + \nabla_2\hat{L}_P\nabla_2)\Phi = 0. \quad (4)$$

Here, it is designated:

$$\hat{L}_T = \hat{L}_T(\omega) = \frac{\partial}{\partial \ell} p \frac{\partial}{\partial \ell} + p\frac{\omega^2}{A^2},$$
$$\hat{L}_P = \hat{L}_P(\omega) = \frac{\partial}{\partial \ell}\frac{1}{p}\frac{\partial}{\partial \ell} + \frac{1}{p}\frac{\omega^2}{A^2}, \quad (5)$$
$$p = \left(\frac{g_2}{g_1}\right)^{1/2}.$$

When deriving equation (4), we have availed ourselves of the transverse small-scale character of the oscillation and have removed the slowly varying term $1/\sqrt{g_3}$ from the operators ∇_1 and ∇_2. Equation (4) is a basic one for the theory to be developed here.

An important role in this theory is played by the quantity p; this especially applies to its dependence on coordinate ℓ. The geometrical meaning of this dependence may be explained in the following way. Let us consider a thin flux tube of rectangular cross-section, having an identically small size $dx^1 = dx^2$ in coordinates x^1 and x^2. The physical size in these same coordinates $\sqrt{g_1}\,dx^1$ and $\sqrt{g_2}\,dx^2$ is, generally speaking, different. The quantity p is their ratio. It is obvious that in a magnetic field with straight field lines it does not change along the field line. In other words, the dependence of p on ℓ is determined by the curvature of the field lines. This dependence was considered in greater detail by Leonovich and Mazur (1990).

The potential Φ determines the wave's electric field, and in terms of the first of Maxwell's equations it also determines the magnetic field. In the main order of perturbation theory, we have:

$$E_1 = -\nabla_1\Phi, \qquad E_2 = -\nabla_2\Phi, \qquad E_3 = 0,$$
$$B_1 = -i\frac{c}{\omega}\frac{1}{p}\nabla_2\frac{\partial\Phi}{\partial\ell}, \qquad B_2 = i\frac{c}{\omega}p\nabla_1\frac{\partial\Phi}{\partial\ell}. \quad (6)$$

As for the component B_3, for it we have:

$$B_3 = i\frac{c}{\omega}\frac{g_3}{\sqrt{g}}(\tilde{\nabla}^j E_j). \quad (7)$$

If we substitute here $E_j = -\nabla_j\Phi$, then we obtain $B_3 = 0$. However, unlike E_3, the component B_3 is zero only in the main order of perturbation theory. In the next order, from equation (3) it follows that:

$$\tilde{P}^{ij}E_j = \hat{L}^{ij}\nabla_j\Phi.$$

By multiplying this relationship from the left by $\tilde{\nabla}_i$, we obtain:

$$\Delta_\perp(\tilde{\nabla}^j E_j) = 2\frac{d\ln(p)}{d\ell}\nabla_1\nabla_2\frac{\partial\Phi}{\partial\ell}, \quad (8)$$

where $\Delta_\perp = \nabla_1^2/g_1 + \nabla_2^2/g_2$ is a two-dimensional Laplacian. The equalities (7) and (8) express B_3 implicitly in terms of Φ.

By virtue of the axial symmetry of the magnetosphere supposed here, the dependence of the potential Φ on the coordinate x^2 can be chosen to be:

$$\Phi(x^1, x^2, x^3) = \tilde{\Phi}(x^1, x^3)\, e^{ik_2x^2},$$

where k_2 is the covariant azimuthal component of the wavevector. If $x^2 = \varphi$, then $k_2 = m$ is the azimuthal wavenumber. It must be stressed again that we are dealing with oscillations with $m \gg 1$. Obviously, with such a dependence on x^2, we have $\nabla_2\Phi = ik_2\Phi$.

3. The boundary condition on the ionosphere

The boundary condition for Alfvén waves on the ionosphere for simple models of the medium was obtained in many papers (Maltsev *et al.*, 1974; Hughes, 1974; Hughes and Southwood, 1976; Alperovich and Federov, 1984; see also reviews by Southwood and Hughes, 1983; Lyatsky and Maltsev, 1983). Within the framework of a more adequate model of the medium for Alfvén waves of a general form, it was obtained in a paper by Leonovich and Mazur (1991). However, the cited papers neglected the existence in the ionosphere of extraneous currents which, according to the scenario presented in the Introduction, have the role of a source for the oscillations of our interest.

The presence of an extraneous current in the ionosphere means that Ohm's law in this medium has the form:

$$\mathbf{j} = \hat{\boldsymbol{\sigma}}\mathbf{E} + \mathbf{j}^{(ext)}. \quad (9)$$

Here $\mathbf{j}$ is the density of current, $\mathbf{E}$ is the wave's electric field, $\hat{\boldsymbol{\sigma}}$ is the conductivity tensor (including longitudinal, Pedersen and Hall conductivities), and $\mathbf{j}^{(ext)}$ is the density of extraneous current. From the relationship (9), it follows that current $\mathbf{j}^{(ext)}$ is associated not with the wave's field, but has some different sources. It seems likely that the most significant excitation mechanism for extraneous current in the frequency range of standing Alfvén waves of our interest ($f \sim 10^{-1}$–10^{-3} Hz) is the motion of neutrals in acoustic–gravity and internal gravity waves which entrains the magnetized electrons and unmagnetized ions in the ionospheric E-region in a different manner. It is also possible that extraneous currents are produced by local electric fields, including those of artificial origin.

The derivation of the boundary condition for Alfvén waves on the ionosphere, in the presence of extraneous currents there, can be accomplished using the same method as used in the cited papers. Without going into details (we intend to have it published in a separate paper), we give only the final result:

$$E_j|_{\ell_\pm} = \mp i\frac{c^2\cos\chi_\pm}{4\pi\omega\Sigma_P^{(\pm)}}\left.\frac{\partial E_j}{\partial\ell}\right|_{\ell_\pm} + \frac{\cos\chi_\pm}{\Sigma_P^{(\pm)}}\nabla_j J_\parallel^{(\pm)}. \quad (10)$$

Here the indices "+" and "−" refer to the ionosphere of the conjugate hemispheres, χ is the angle between the field line and the vertical to the ionosphere at the point of their intersection, $\Sigma_P^{(\pm)}$ is the integral Pedersen conductivity of the ionosphere, and $J_\parallel^{(\pm)}$ is a function, satisfying the equation:

$$\Delta_\perp^{(\pm)}J_\parallel^{(\pm)} = j_\parallel^{(\pm)}, \quad (11)$$

where:

$$\Delta_\perp^{(\pm)} = \Delta_\perp|_{\ell_\pm} \equiv \frac{1}{g_1^{(\pm)}}\nabla_1^2 + \frac{1}{g_2^{(\pm)}}\nabla_2^2,$$

and $j_\parallel^{(\pm)}$ is the density of extraneous field-aligned current at the ionosphere–magnetosphere interface (we shall consider the direction towards increasing coordinate ℓ, i.e. from ℓ_- to ℓ_+ to be the positive direction for it in both hemispheres).

If the relationship $E_j = -\nabla_j\Phi$ is substituted into equation (10), then we obtain the boundary condition for the function Φ:

$$\Phi|_{\ell_\pm} = \mp i\frac{v_\pm}{\omega}\frac{\partial\Phi}{\partial\ell}\bigg|_{\ell_\pm} - \frac{J_\parallel^{(\pm)}}{V_\pm}. \qquad (12)$$

Here, it is designated:

$$v_\pm = \frac{c^2\cos\chi_\pm}{4\pi\Sigma_P^{(\pm)}}, \qquad V_\pm = \frac{\Sigma_P^{(\pm)}}{\cos\chi_\pm} \equiv \frac{c^2}{4\pi v_\pm}.$$

The parameters $v_\pm$ and $V_\pm$ have the dimensions of velocity. On the order of magnitude $v_\pm \sim 10^2$ km s^{-1} and $V_\pm \sim 10^8$ km s^{-1} for the dayside ionosphere, and $v_\pm \sim 10^3$ km s^{-1} and $V_\pm \sim 10^7$ km s^{-1} for the nightside ionosphere. The first term on the right-hand side of equation (12) describes the dissipation of the waves in the ionosphere, and the second term represents its generation by extraneous currents. We shall consider both these effects to be small, that is, it will be assumed that the parameter $v_\pm$ is small and the parameter $V_\pm$ is large. The precise meaning of this assumption will be formulated below. In developing the perturbation theory using the above parameters, in the main order we shall assume that:

$$\Phi|_{\ell_\pm} = 0. \qquad (13)$$

4. Structure of the oscillations in transverse coordinate

The transverse small-scale character of the oscillations under investigation makes using the WKB approximation quite natural. By choosing the dependence on the coordinate x^2 in the form of one azimuthal harmonic, we assume that:

$$\Phi = \exp(iQ + ik_2x^2),$$

where $Q = Q(x^1, \ell)$ is the quasiclassical phase. The condition for the small-scale character of the potential in coordinate x^1:

$$\left|\frac{1}{\sqrt{g_1}}\frac{\partial\Phi}{\partial x^1}\right| \gg \left|\frac{\partial\Phi}{\partial\ell}\right|,$$

yields an analogous inequality for phase Q:

$$\left|\frac{1}{\sqrt{g_1}}\frac{\partial Q}{\partial x^1}\right| \gg \left|\frac{\partial Q}{\partial\ell}\right|,$$

that is, the dependence of the function Q on the coordinate x^1 is much stronger than on the coordinate ℓ. This means that from it one can separate the main term which depends only on x^1. In other words, when expanding the phase as an asymptotic series of the WKB approximation:

$$Q = Q_0 + Q_1 + Q_2 + \ldots,$$

the term of the main order Q_0 can be considered independent of the coordinate ℓ. We designate:

$$\psi = Q_0; \ \exp[i(Q_1 + Q_2 + \ldots)] = H + h + \ldots,$$

or, in other words, we represent the WKB approximation as:

$$\Phi(x^1, x^2, \ell, \omega) = \exp[i\psi(x^1, \omega) + ik_2x^2][H(x^1, \ell, \omega) + h(x^1, \ell, \omega) + \ldots]. \qquad (14)$$

Note also that the function:

$$\tilde\psi(x^1, x^2, \omega) = \psi(x^1, \omega) + k_2x^2,$$

represents a full quasiclassical phase.

Quasiclassical covariant components of the wavevector are known to be defined by the relationships:

$$k_1 = \partial\tilde\psi/\partial x^1, \qquad k_2 = \partial\tilde\psi/\partial x^2.$$

From the former equality, it follows that:

$$k_1 = k_1(x^1, \omega) = \partial\psi(x^1, \omega)/\partial x^1,$$

and the latter, as one would expect, yields the identity $k_2 = k_2$.

Despite the fact that k_1 and k_2 do not depend on ℓ, the square of the wavevector:

$$k_\perp^2 = k_1^2/g_1 + k_2^2/g_2,$$

does depend on ℓ.

Substituting the expression (14) into equation (4), in the main order of the WKB approximation we obtain:

$$\hat L H = 0, \qquad (15)$$

where:

$$\hat L = \hat L(x^1, k^1, k^2, \omega) \equiv k_1^2\hat L_T + k_2^2\hat L_P = \frac{\partial}{\partial\ell}q\frac{\partial}{\partial\ell} + q\frac{\omega^2}{A^2},$$

$$q = \sqrt{g_\perp}k_\perp^2 = pk_1^2 + p^{-1}k_2^2.$$

In the same main order, we take the boundary condition in the form of equation (13), that is, we put:

$$H|_{\ell_\pm} = 0. \qquad (16)$$

At given values of x^1 and ω, the relationships (15) and (16) can be considered as the problem for eigenvalues for the quantity $\kappa = k_1/k_2$. One can make sure that such a treatment is possible by representing these relationships as:

$$(\kappa^2\hat L_T + \hat L_P)H = 0; \qquad H|_{\ell_\pm} = 0. \qquad (17)$$

Let:

$$\kappa = \kappa_N(x^1, \omega), \qquad H = H_N(x^1, \ell, \omega), \qquad (18)$$

be the problem solutions for eigenvalues. Here, $N = 1, 2, \ldots$ is the harmonic number equal to the number of half-waves of the function H_N on a field line. At a given value of k_2, equations (15) and (16) can be treated as a problem for eigenvalues for k_1. It is clear that:

$$k_1 = k_{1N}(x^1, \omega) = k_2\kappa_N(x^1, \omega). \qquad (19)$$

From this, we have:

$$\psi = \psi_N(x^1, \omega) = \int k_{1N}(x^1, \omega)\,dx^1 = k_2 \int \kappa_N(x^1, \omega)\,dx^1. \tag{20}$$

Thus, the solution of the longitudinal problem for eigenvalues defines the main order of the WKB approximation in transverse coordinate x^1.

To conclude this section, we wish to give formulas for the disturbed electric and magnetic fields in the WKB approximation. Substituting equation (14) into equations (6), (7) and (8), in the main order we have:

$$E_1 = -ik_1 H e^{i\tilde{\psi}}, \qquad E_2 = -ik_2 H e^{i\tilde{\psi}}, \qquad E_3 = 0,$$

$$B_1 = \frac{c}{\omega}\frac{k_2}{p}\frac{\partial H}{\partial \ell} e^{i\tilde{\psi}}, \qquad B_2 = -\frac{c}{\omega} k_1 p \frac{\partial H}{\partial \ell} e^{i\tilde{\psi}}, \tag{21}$$

$$B_3 = -2i\frac{c}{\omega}\sqrt{g_3}\frac{k_1 k_2}{q}\frac{\partial \ln p}{\partial \ell}\frac{\partial H}{\partial \ell} e^{i\tilde{\psi}}.$$

5. A qualitative investigation of the longitudinal problem for eigenvalues

The problem (15), (16) defines H_N as a function of ℓ up to within an arbitrary factor. In order to fix this factor subsequently, we shall consider the normalized solutions of this problem $R_N(x^1, \ell, \omega)$ by the relationships:

$$\hat{L}(x^1, k_{1N}, k_2, \omega) R_N = 0, \qquad R_N|_{\ell_\pm} = 0;$$
$$\oint \frac{q_N}{A^2} R_N^2\, d\ell = 1. \tag{22}$$

Here:

$$q_N = pk_{1N}^2 + p^{-1}k_2^2 = k_2^2(p\kappa_N^2 + p^{-1}),$$

and the line contour integral over the closed contour means integration along the field line "there and back" between the magneto-conjugate magnetospheres.

The relationship $k_1/k_2 = \kappa(x^1, \omega)$ at a given value of x^1 defines the functional connection between the frequency ω and the parameter $\kappa = k_1/k_2$. Let us introduce an inverse function:

$$\omega = \omega(x^1, k_1/k_2). \tag{23}$$

Values of ω_N can be regarded as the solution of the problem for eigenvalues (15), (16) for the parameter ω at given values of the parameters k_1 and k_2, and the equality (23) as a local dispersion equation. Eigenfunctions corresponding to such a problem statement are $R_N[x^1, \ell, \omega_N(x^1, k_1/k_2)]$.

A special role in a subsequent discussion is played by two limiting cases: $\kappa = 0$ and $\kappa = \infty$, corresponding to poloidal and toroidal modes. Let us consider them in greater detail.

When $\kappa = 0$ ($k_1 = 0$), the problem for eigenvalues (15), (16) takes the form:

$$\hat{L}_P(\omega)H = 0; \qquad H|_{\ell_\pm} = 0.$$

Its solutions will be designated as:

$$\omega = \Omega_N^P(x^1), \qquad H = P_N(x^1, \ell)$$

and will be called the poloidal eigenfrequencies and the poloidal eigenfunctions, respectively. The latter will be considered normalized by the condition:

$$\oint \frac{1}{pA^2} P_N^2\, d\ell = 1.$$

It is easy to see that:

$$\Omega_N^P(x^1) = \omega_N(x^1, 0), \qquad P_N(x^1, \ell) = k_2 R_N[x^1, \ell, \Omega_N^P(x^1)].$$

When $\kappa \to \infty$ ($k_1 \to \infty$), from equations (15) and (16) we have:

$$\hat{L}_T(\omega)H = 0, \qquad H|_{\ell_\pm} = 0.$$

The solutions of this problem—toroidal eigenfrequencies and eigenfunctions—will be designated as:

$$\omega = \Omega_N^T(x^1), \qquad H = T_N(x^1, \ell).$$

We shall assume the normalization condition

$$\oint \frac{p}{A^2} T_N^2\, d\ell = 1.$$

We then get:

$$\Omega_N^T = \omega_N(x^1, \infty), \qquad T_N(x^1, \ell) = k_1 R_N[x^1, \ell, \Omega_N^T(x^1)].$$

The last equality should be understood so that when $k_1 \to \infty$, the function $R_N = T_N/k_1$ tends to zero.

For the theory we develop here, a crucial role is played by the difference of the poloidal and toroidal frequencies. Their difference $\Delta\Omega_N = \Omega_N^T - \Omega_n^P$ is called the polarization splitting of the spectrum. A useful analytic expression can be obtained for it. We proceed from the identities:

$$\frac{\partial}{\partial \ell} p \frac{\partial T_N}{\partial \ell} + p\frac{\Omega_N^{T^2}}{A^2} T_N = 0, \qquad \frac{\partial}{\partial \ell}\frac{1}{p}\frac{\partial P_N}{\partial \ell} + \frac{1}{p}\frac{\Omega_N^{P^2}}{A^2} P_N = 0.$$

We multiply the first of them by P_N/p and the second, by pT_N, extract one from the other, and integrate along the field line. Upon transforming, by means of integration by parts, we obtain the equality:

$$\Omega_N^{T^2} - \Omega_N^{P^2} = \oint \frac{\partial^2 \ln p}{\partial \ell^2} P_N T_N\, d\ell \Big/ \oint \frac{1}{A^2} P_N T_N\, d\ell. \tag{24}$$

From equation (24), it follows that the polarization splitting is due to the curvature of the geomagnetic field lines. Indeed, in a magnetic field with straight field lines, the value of $p = (g_2/g_1)^{1/2}$ does not depend on ℓ and, therefore, the right-hand side of equation (24) is zero. In the next section it will be shown that the difference $\Delta\Omega_N$ is small compared with the frequencies themselves.

For the mode with a given frequency ω, a special role is played by magnetic surfaces defined (at a given N) by the equations:

$$\Omega_N^P(x^1) = \omega, \qquad \Omega_N^T(x^1) = \omega. \tag{25}$$

Let us call them the poloidal and toroidal resonance surfaces and let us denote their coordinates, that is, the solutions of equations (25), respectively, by $x^1 = x_{PN}^1(\omega)$ and $x^1 = x_{TN}^1(\omega)$. The distance between them can be characterized by the difference $\Delta x_N^1 = x_{TN}^1 - x_{PN}^1$. If $\Delta\Omega_N > 0$, then with a monotonically decreasing function Ω_N^P and

Ω_N^T, the poloidal surface lies inside the toroidal surface, $x_{TN}^1 > x_{PN}^1$; otherwise, $x_{TN}^1 < x_{PN}^1$. Taking into account that $\Delta\Omega_N \ll \Omega_N^P, \Omega_N^T$, it is easy to obtain an explicit expression for the quantity Δx_N^1. Let the functions $\Omega_N^{P,T}(x^1)$ near the resonance surfaces be represented as:

$$\Omega_N^P = \omega\left(1 - \frac{x^1 - x_{PN}^1}{2\ell_N}\right), \qquad \Omega_N^T = \omega\left(1 - \frac{x^1 - x_{TN}^1}{2\ell_N}\right). \tag{26}$$

Here ℓ_N is a typical scale of variation of the functions $\Omega_N^{P,T}(x^1)$ which, when $\Delta\Omega_N \ll \Omega_N^{P,T}$, can be considered equal for both functions. This inequality suggests that the validity ranges of the expansions (26) overlap. By extracting one from the other, we obtain:

$$\Delta x_N^1 = 2\alpha_N \ell_N, \tag{27}$$

where it is designated $\alpha_N = \Delta\Omega_N/\omega$ and it is assumed that $\alpha_N \ll 1$.

From the definitions of resonance surfaces it follows that on a poloidal surface the function $k_{1N}(x^1, \omega)$ goes to zero, and on a toroidal surface it extends to infinity. We investigate its behaviour in the vicinities of these surfaces. For this purpose, we make use of perturbation theory. Near the poloidal surface, when $|x^1 - x_{PN}^1| \ll \Delta x_N^1$, values of $\omega^2 - \Omega_N^{P^2}$ and k_{1N}^2 can be considered small. Put also $H = P_N + h_N$, where h_N is a small correction. By linearizing the problem (15), (16) in small values, we have:

$$k_2^2 \hat{L}_P(\Omega_N^P) h_N + k_{1N}^2 \hat{L}_T(\Omega_N^P) P_N + k_2^2 \frac{\omega^2 - \Omega_N^{P^2}}{pA^2} P_N = 0,$$

$$h_N|_{\ell_\pm} = 0.$$

We multiply this equation by P_N and integrate along the field line. Taking into account the Hermitian character (together with the boundary condition) of the operator $\hat{L}_P$, we obtain:

$$k_{1N}^2 = k_2^2 \frac{\omega^2 - \Omega_N^{P^2}}{w_N^P},$$
$$w_N^P = -\oint P_N \hat{L}_T(\Omega_N^P) P_N \, d\ell = \oint \frac{\partial^2 P}{\partial \ell^2} P_N^2 \, d\ell. \tag{28}$$

In much the same way, near the toroidal surface, when $|x^1 - x_{TN}^1| \ll \Delta x_N^1$:

$$k_{1N}^2 = -k_2^2 \frac{w_N^T}{\omega^2 - \Omega_N^{T^2}},$$
$$w_N^T = \oint T_N \hat{L}_P(\Omega_N^T) T_N \, d\ell = -\oint \frac{\partial^2 p^{-1}}{\partial \ell} T_N^2 \, d\ell. \tag{29}$$

Analytic estimates and results of numerical calculations given in the next section show that the constants w_N^P and w_N^T are positive.

If the expansions (26) are applicable, then from equations (28) and (29) we have, respectively:

$$k_{1N}^2 = k_2^2 \frac{\omega^2}{w_N^P} \frac{x^1 - x_{PN}^1}{\ell_N}, \qquad k_{1N}^2 = -k_2^2 \frac{w_N^T}{\omega^2} \frac{\ell_N}{x^1 - x_{TN}^1}. \tag{30}$$

From this it is evident that the poloidal resonance surface is, in coordinate x^1, a usual turning point, at which k_1^2 goes to zero, and the toroidal resonance surface is a singular turning point where k_1^2 has a pole. Near the poloidal surface, the transparency region lies at $x^1 > x_{PN}^1$, and near the toroidal surface it lies at $x^1 < x_{TN}^1$, i.e. the transparency region of the mode lines in the range $x_{PN}^1 < x^1 < x_{TN}^1$.

Opacity regions where values of k_1^2 are negative lie outside this range. Asymptotic values of k_{1N}^2 in these regions can be determined analytically. We shall not explain here the procedure of the derivation, but give the result:

$$k_{1N}^2 \to \begin{cases} -k_2^2/p_{\min}^2, & x^1 - x_{TN}^1 \gg \Delta x_N^1, \\ -k_2^2/p_{\max}^2, & x_{PN}^1 - x^1 \gg \Delta x_N^1, \end{cases} \tag{31}$$

where $p_{\min}$ and $p_{\max}$ are, respectively, the minimum and maximum values of the function $p = p(\ell)$ on the field line. In the simplest (dipole-field, for example) models of the geomagnetic field the $p_{\min}$—and $p_{\max}$—values are reached, respectively, on the equator and on the ionosphere.

In closing this section, we consider the question of the transverse group velocity of the oscillations being studied. We define its contravariant components in the usual way: $v_N^i = \partial\omega_N/\partial k_i$. We have:

$$v_N^1 = \frac{\partial\omega_N(x^1, k_1/k_2)}{\partial k_1} = \frac{1}{k_2} \frac{\partial\omega_N(x^1, \kappa)}{\partial\kappa}$$
$$= \frac{1}{k_2}\left(\frac{\partial\kappa_N}{\partial\omega}\right)^{-1} = \left(\frac{\partial k_{1N}}{\partial\omega}\right)^{-1}. \tag{32}$$

In order to obtain the formula for $\partial k_{1N}/\partial\omega$, we differentiate equation (22) with respect to ω. Taking into consideration that:

$$\frac{\partial\hat{L}}{\partial\omega} = 2k_{1N} \frac{\partial k_{1N}}{\partial\omega} \hat{L}_T(\omega) + 2\omega \frac{q_N}{A^2},$$

we obtain the equation:

$$\hat{L}\frac{\partial R_N}{\partial\omega} + 2k_{1N} \frac{\partial k_{1N}}{\partial\omega} \hat{L}_T(\omega) R_N + 2\omega \frac{q_N}{A^2} R_N = 0.$$

We multiply it by R_N and integrate along the field line. In view of the Hermitian character of the operator $\hat{L}$ and the normalization condition (22), we get:

$$v_N^1 = \left(\frac{\partial k_{1N}}{\partial\omega}\right)^{-1} = -\frac{k_{1N}}{\omega}\oint R_N \hat{L}_T(\omega) R_N \, d\ell$$
$$= \frac{k_{1N}}{\omega}\oint p\left[\left(\frac{\partial R_N}{\partial\ell}\right)^2 - \frac{\omega^2}{A^2} R_N^2\right] d\ell. \tag{33a}$$

Similarly, we obtain the equality:

$$v_N^2 = -\frac{k_2}{\omega}\oint R_N \hat{L}_P(\omega) R_N \, d\ell$$
$$= \frac{k_2}{\omega}\oint \frac{1}{p}\left[\left(\frac{\partial R_N}{\partial\ell}\right)^2 - \frac{\omega^2}{A^2} R_N^2\right] d\ell. \tag{33b}$$

Using the relationship:

$$\frac{\omega^2}{A^2}R_N^2 = -\frac{1}{q_N}R_N\frac{\partial}{\partial \ell}q_N\frac{\partial R_N}{\partial \ell},$$

we bring the expressions for group velocities into the form:

$$v_N^i = \frac{\tilde{k}^i k_1 k_2}{\omega}\oint\left(\frac{p'/p}{q_N}\right)' R_N^2\,\mathrm{d}\ell.$$

Remember that $\tilde{k}^1 = k_2$ and $\tilde{k}^2 = -k_1$.

In the vicinities of the resonance surfaces these formulas simplify. Near the poloidal surface:

$$v_N^1 = \frac{k_{1N}w_N^{\mathrm{P}}}{\omega k_2^2}, \qquad v_N^2 = -\frac{k_{1N}^2 w_N^{\mathrm{P}}}{\omega k_2^3} \tag{34}$$

and near the toroidal surface:

$$v_N^1 = \frac{k_2^2 w_N^{\mathrm{T}}}{\omega k_{1N}^3}, \qquad v_N^2 = -\frac{k_2 w_N^{\mathrm{T}}}{\omega k_{1N}^2}. \tag{35}$$

From this and from the relationship (30) it is evident that the group velocity goes to zero on the poloidal surface by the law:

$$v_N^1 \sim (x^1 - x_{\mathrm{P}N}^1)^{1/2}, \qquad v_N^2 \sim x^1 - x_{\mathrm{P}N}^1 \tag{36}$$

and on the toroidal surface by the law:

$$v_N^1 \sim (x_{\mathrm{T}N}^1 - x^1)^{3/2}, \qquad v_N^2 \sim x_{\mathrm{T}N}^1 - x^1. \tag{37}$$

Formulas (33a,b) directly yield the equality:

$$k_{1N}v_N^1 + k_2 v_N^2 = 0, \tag{38}$$

which means that the transverse group velocity v_N^i is perpendicular to the phase gradient $k_i = \nabla_i\psi$, i.e. is directed along the characteristic, the line of constant phase (see Fig. 2). Making use of the notion of group velocity one can introduce a new variable τ, the time taken by the wave to propagate along the characteristic. Put:

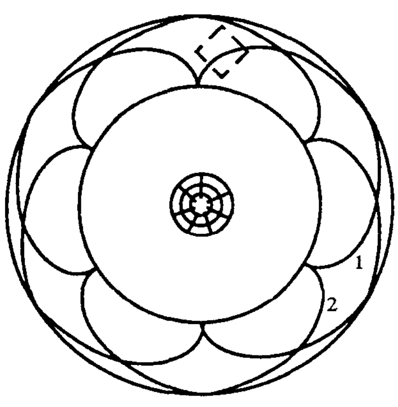

Fig. 2. Constant phase lines $\Psi(x^1, x^2) = $ const. (characteristics). Curves 1 correspond to values of $k_2 > 0$, and curves 2 refer to $k_2 < 0$. Concentric circles are the cross-sections of resonance surfaces: inner—poloidal, outer—toroidal. The characteristics are normal to the poloidal surface and are tangent to the toroidal surface. The group velocity of the waves is directed along the characteristic. The square corresponds to a possible position of the observation area on the STARE radar (Walker *et al.*, 1982) projected along field lines onto the equatorial plane

$$\mathrm{d}\tau = \frac{\mathrm{d}x^1}{v_N^1} = \frac{\mathrm{d}x^2}{v_N^2}, \tag{39}$$

where it is assumed that the differentials of the coordinates $\mathrm{d}x^1$ and $\mathrm{d}x^2$ are taken along the characteristic, i.e. are related by the relation $k_1\mathrm{d}x^1 + k_2\mathrm{d}x^2 = 0$. Let us make the convention that time τ is reckoned from the poloidal resonance surface. Then:

$$\tau = \int_{x_{\mathrm{P}N}^1}^{x^1}\frac{\mathrm{d}x^{1'}}{v_N^1}. \tag{40}$$

Using the relationships (36) and (37), it is easy to make sure that the integral (40) converges on the lower limit and diverges when $x^1 \to x_{\mathrm{T}N}^1$. This latter means that the time taken by the wave to approach the toroidal surface is infinite.

From the definition (39) it follows that:

$$v_N^1 = \frac{\mathrm{d}x^1}{\mathrm{d}\tau}, \qquad v_N^2 = \frac{\mathrm{d}x^2}{\mathrm{d}\tau},$$

where the differentials of the coordinates are also taken along the characteristic. Let the full time derivative of the function of coordinates x^1, x^2 be defined by the equality:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = v_N^1\frac{\partial}{\partial x^1} + v_N^2\frac{\partial}{\partial x^2}. \tag{41}$$

The relationship (38) can then be represented as:

$$\frac{\partial\tilde{\psi}}{\partial\tau} = 0,$$

i.e. the phase is constant along the characteristic.

6. Solving the longitudinal problem in the WKB approximation and numerically

The solution of the longitudinal problem (22) for any realistic models of the geomagnetic field and plasma is, undoubtedly, a numerical problem. Only for harmonics with large numbers N can the analytical method, the WKB approximation in longitudinal coordinate ℓ, be used. In this section we shall make use of both methods and shall compare their results. Let us start with the WKB method.

We assume $R_N(\ell) = \exp[\mathrm{i}s(\ell)]$, where $s(\ell)$ is a large quasiclassical phase. Equation (22) takes the form:

$$-s'^2 + \mathrm{i}s'' + \mathrm{i}(\ln q)'s' + \omega^2/A^2 = 0.$$

Here and later, the prime denotes the derivative with respect to ℓ. As will be shown later, in order to obtain the result to the required accuracy, in the asymptotic expansion of phase s it is necessary to retain three terms:

$$s = s_0 + s_1 + s_2 + \ldots.$$

In the main (zeroth) order, we have:

$$-s_0'^2 + \frac{\omega^2}{A^2} = 0; \qquad s_0(\ell) = \pm\omega\int\frac{\mathrm{d}\ell}{A(\ell)}.$$

In the next (first) order:

$$-2s_0's_1'+\mathrm{i}s_0''+\mathrm{i}(\ln q)'s_0' = 0,$$

from which we get:

$$s_1 = -\mathrm{i}\ln C+\frac{\mathrm{i}}{2}\ln\frac{q}{A}, \qquad \mathrm{e}^{\mathrm{i}s_1} = C\left(\frac{A}{q}\right)^{1/2},$$

where C is a constant. In the second order:

$$-2s_0's_2'-s_1'^2+\mathrm{i}s_1''+\mathrm{i}(\ln q)'s_1' = 0.$$

A little manipulation yields:

$$s_2 = \pm\frac{1}{8\omega}\int A\,[(\ln A)'^2-(\ln q)'^2+2(\ln A)''-2(\ln q)'']\,\mathrm{d}\ell'.$$

We designate $\bar{s} = s_0+s_2$. A general solution can then be written as:

$$R_N = \left(\frac{A}{q}\right)^{1/2}(c_+\mathrm{e}^{\mathrm{i}\bar{s}}+c_-\mathrm{e}^{-\mathrm{i}\bar{s}}) \equiv \left(\frac{A}{q}\right)^{1/2}(c_1\sin\bar{s}+c_2\cos\bar{s}).$$

The boundary condition (22) gives $c_2 = 0$ and leads to the quantization condition:

$$\omega\oint\frac{\mathrm{d}\ell}{A}+\frac{1}{8\omega}\oint A\,[(\ln A)'^2-(\ln q)'^2+2(\ln A)''-2(\ln q)'']\,\mathrm{d}\ell = 2\pi N.$$

Assuming number N to be large and by solving this equation using the iteration method, we obtain:

$$\omega = \omega_N \equiv \frac{2\pi N}{t_A}-\frac{1}{16\pi N}\oint A\,[(\ln A)'^2-(\ln q)'^2+2(\ln A)''-2(\ln q)'']\,\mathrm{d}\ell, \quad (42)$$

where:

$$t_A = t_A(x^1) = \oint\frac{\mathrm{d}\ell}{A(x^1,\ell)},$$

is the transit time with a local Alfvén velocity along the field line "there and back". Since:

$$(\ln q)' = \frac{p^2\kappa^2-1}{p^2\kappa^2+1}(\ln p)', \quad (43)$$

then the equality (42) defines $\omega = \omega_N$ as a function of the parameter κ. It can also be regarded as the equation, defining the function $\kappa = \kappa_N(\omega)$. From the relationship (42) it is evident that the mode dispersion is manifest only in the second order of the WKB approximation. As far as the eigenmodes are concerned, subsequently it will be sufficient for us to limit ourselves to two orders. Using the normalization condition (22) we define the constant C_1 to give:

$$R_N = \left(\frac{2A}{qt_A}\right)^{1/2}\sin\left(\frac{2\pi N}{t_A}\int_{\ell_-}^{\ell}\frac{\mathrm{d}\ell'}{A}\right). \quad (44)$$

From the equalities (42) and (43), the formulas:

$$\Omega_N^{\mathrm{P}} = \frac{2\pi N}{t_A}+\frac{1}{16\pi N}\oint A\,[(\ln p)'^2-(\ln A)'^2-2(\ln p)''-2(\ln A)'']\,\mathrm{d}\ell, \quad (45a)$$

$$\Omega_N^{\mathrm{T}} = \frac{2\pi N}{t_A}+\frac{1}{16\pi N}\oint A\,[(\ln p)'^2-(\ln A)'^2+2(\ln p)''-2(\ln A)'']\,\mathrm{d}\ell, \quad (45b)$$

follow as partial cases, and from equation (44) we have:

$$P_N = \left(\frac{2pA}{t_A}\right)^{1/2}\sin\left(\frac{2\pi N}{t_A}\int_{\ell_-}^{\ell}\frac{\mathrm{d}\ell'}{A}\right),$$
$$T_N = \left(\frac{2A}{pt_A}\right)^{1/2}\sin\left(\frac{2\pi N}{t_A}\int_{\ell_-}^{\ell}\frac{\mathrm{d}\ell'}{A}\right). \quad (46)$$

For the polarizational splitting of spectrum, from equations (45a,b) we get:

$$\Delta\Omega_N = \frac{1}{4\pi N}\oint A\,(\ln p)''\,\mathrm{d}\ell. \quad (47)$$

This same formula is obtainable from the equality (24) by substituting the expressions (46) into it. Proceeding in the same manner, from equations (28) and (29), we have:

$$w_N^{\mathrm{P}} = \frac{1}{t_A}\oint App''\,\mathrm{d}\ell, \qquad w_N^{\mathrm{T}} = -\frac{1}{t_A}\oint A\frac{1}{p}\left(\frac{1}{p}\right)''\mathrm{d}\ell. \quad (48)$$

Note that the expressions (28), (29) for k_{1N}^2 can be obtained from equation (42), by expanding it, in the first case, in the small parameters $\omega-\Omega_N^{\mathrm{P}}$ and κ^2, and in the second case, in the parameters $\omega-\Omega_N^{\mathrm{T}}$ and κ^{-2}. In this case, constants $w_N^{\mathrm{P,T}}$ are obtained immediately in the form of equation (48). Finally, we give, in the approximation under consideration, the formula for the group velocity:

$$v_N^i = \frac{\bar{k}^i k_1 k_2}{2\pi N}\oint\left(\frac{p'/p}{q}\right)'\frac{A}{q}\,\mathrm{d}\ell. \quad (49)$$

Let us now describe the results of numerical calculations. The goal of these calculations is to test and illustrate the above general considerations. In accordance with this goal, we have limited our attention to a relatively simple model of the magnetosphere. The geomagnetic field was assumed to be a dipole field. The equation for field lines in this case has the form $r = a\cos^2\theta$, where r is the radius vector of a point on the field line, θ is the geomagnetic latitude of this point, and a is the equatorial radius of the field line. Put $x^1 = a$, and $x^2 = \varphi$. We can then demonstrate that:

$$g_1 = \cos^6\theta\,(1+3\sin^2\theta)^{-1}, \qquad g_2 = a^2\cos^6\theta$$

and, consequently:

$$p = a(1+3\sin^2\theta)^{1/2}. \quad (50)$$

A length element of the field line is given by the expression:

$$\mathrm{d}\ell = a\cos\theta(1+3\sin^2\theta)^{1/2}\,\mathrm{d}\theta.$$

Points of intersection of the field line with the ionospheres of the conjugate hemispheres have latitudes $\theta_+ = \theta_*$ and $\theta_- = -\theta_*$, where:

$$\theta_* = \arccos(R_1/a)^{1/2}.$$

and R_I is the radius of the upper boundary of the ionosphere (reckoned from the Earth's centre). Leonovich and Mazur (1991) gave a detailed explanation that this boundary should be chosen at 1000–2000 km altitude above the ground, i.e. where a rapid growth in Alfvén velocity with height discontinues. Using in the meridional plane the variables a and θ (of course, they are not orthogonal ones) as the coordinates, the geomagnetic field modulus can be represented as:

$$B(a,\theta) = B_0(a_0/a)^3\beta(\theta);$$

$$\beta(\theta) = (1+3\sin^2\theta)^{1/2}\cos^{-6}\theta.$$

Here B_0 is the value of geomagnetic field in the equatorial plane on some separated magnetic shell $a = a_0$. The Alfvén velocity distribution was modelled by an expression of the form:

$$A(a,\theta) = A_0(a_0/a)^{\mu}[\beta(\theta)]^{\nu}; \qquad (51)$$

by fitting numerical values of the constants A_0, μ and ν, this expression can be used to approximate a wide class of Alfvén velocity distributions in the magnetosphere. Yet, of course, the expression (51) has limited capabilities. Thus, owing to the monotonic dependence on a, it cannot model the plasmapause.

We shall verify the numerical calculations for the following values of constant $a_0 = 4R_E = 2.5\times10^4$ km, $R_I = 7.9\times10^3$ km, $A_0 = 10^3$ km s^{-1}, $\mu = 3/2$ and $\nu = 1/4$. Results of these calculations are presented in Figs 3–6. Let us comment on them.

In Fig. 3, an interesting result is the multiple decrease of the value of $\Delta\Omega_N$ in going from $N = 1$ to $N = 2$. In view of the formula of the WKB approximation [equation (47)], one would expect a decrease by several times only.

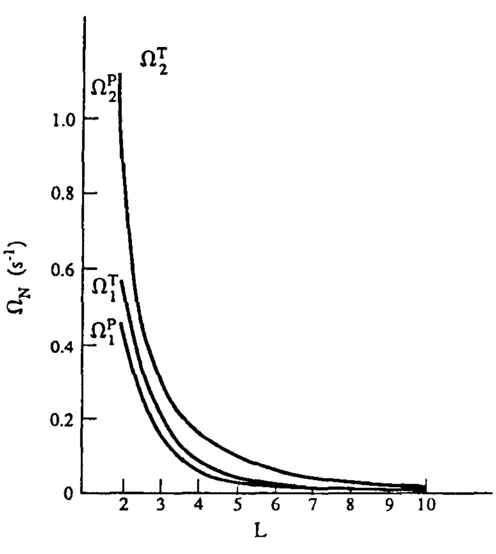

Fig. 3. The dependence of the poloidal Ω_N^P and the toroidal Ω_N^T on eigenfrequencies for $N = 1, 2$ on the McIlwain parameter $L = a/R_E$. The splitting of the frequencies $\Delta\Omega_N = \Omega_N^T - \Omega_N^P$ is relatively large for $N = 1$ ($\eta = \Delta\Omega_1/\Omega_1^P = 0.2$ for $L = 2$; $\eta = 0.35$ for $L = 6$) and decreases abruptly with increasing N: for $N = 2$ ($\eta = \Delta\Omega_2/\Omega_2^P = 0.018$ for $L = 2$; $\eta = 0.005$ for $L = 6$), becoming negligibly small for $N > 3$

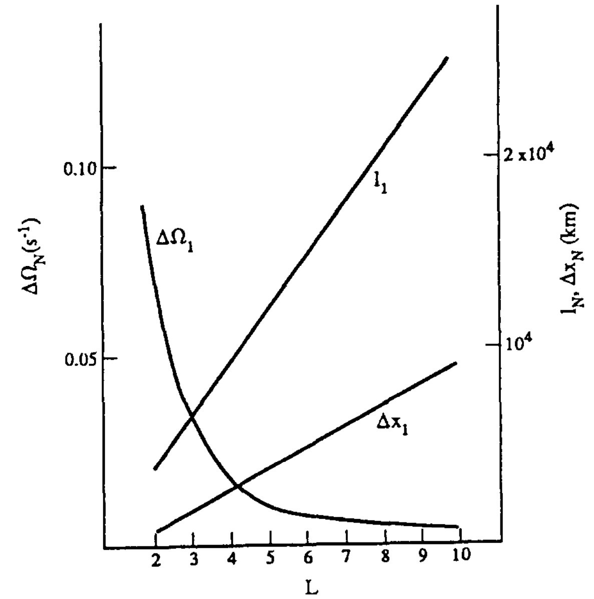

Fig. 4. The dependence of $\Delta\Omega_N$, ℓ_N and Δx_N that characterize the mode $N = 1$, on the McIlwain parameter, L

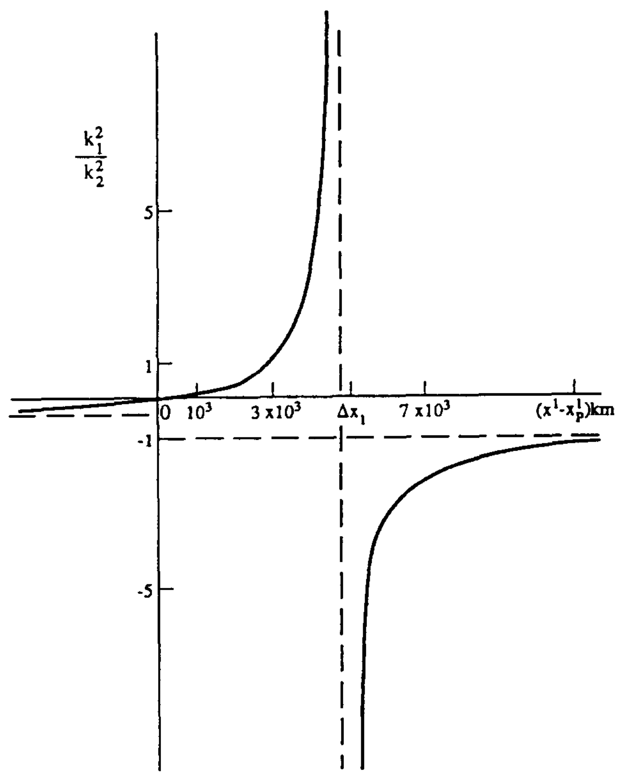

Fig. 5. The plot of the dimensionless ratio $k_{1N}^2 a^2/m^2$ vs the radial coordinate $x^1 \equiv a$ reckoned from the resonance surface x_{PN}^1 for the fundamental harmonic $N = 1$. The point x_{PN}^1 lies on the magnetic shell $L = 6$, which corresponds to the frequency of the mode $\omega = 0.018$ s^{-1} (i.e. $f = \omega/2\pi = 0.03$ Hz)

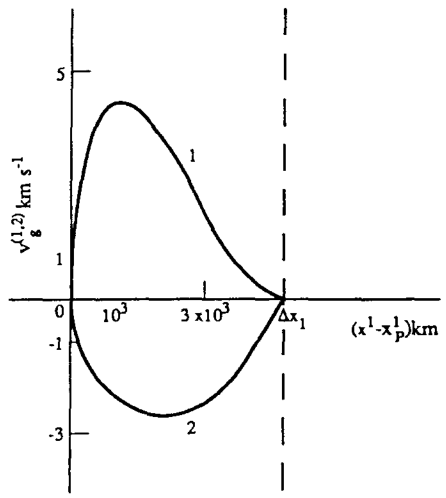

Fig. 6. The plot of group velocities v_g^1 and v_g^2 vs the radial coordinate $x^1 \equiv a$, lying on the same magnetic shell as in Fig. 5

However, the WKB approximation is, strictly speaking, inapplicable for fundamental modes, but it usually also gives correct (on the order of magnitude) results for them. Incidentally, this result depends markedly on the chosen model for the shape of geomagnetic field lines. If we use a more general form of the equation for field lines $r = a\cos^\alpha\theta$, where $\alpha = 2$ corresponds to the dipole field, then it turns out that when $\alpha < 2$ the ratio $\eta = \Delta\Omega_1/\Delta\Omega_2$ is larger, and when $\alpha > 2$, is smaller than in the dipole field. This ratio is minimal on inner magnetic shells ($\eta \approx 2.5$ for $\alpha = 1.25$ and $\eta \approx 4$ for $\alpha = 2$ when $L = 2$), and increases markedly at transition into the outer magnetosphere ($\eta \approx 4.5$ for $\alpha = 1.25$ and $\eta \approx 25$ for $\alpha = 2$ when $L = 6$). Thus, we arrive at an important conclusion, namely a large value of the difference $\Delta\Omega_N$, which is needed for a successful application of the theory concerned, is always satisfied for $N = 1$, but may not hold for $N \geqslant 2$.

We have defined the value of ℓ_N given in Fig. 4 (for $N = 1$) as:

$$\frac{1}{2\ell_N} = \frac{\mathrm{d}\ln(\Omega_N^{\mathrm{P}}\Omega_N^{\mathrm{T}})^{1/2}}{\mathrm{d}x^1} = \frac{1}{2}\left(\frac{\mathrm{d}\ln\Omega_N^{\mathrm{P}}}{\mathrm{d}x^1} + \frac{\mathrm{d}\ln\Omega_N^{\mathrm{T}}}{\mathrm{d}x^1}\right), \quad (52)$$

on the order of magnitude $\ell_N \sim a$. It depends weakly on number N. On the contrary, the value of $\Delta x_N = 2\alpha_N\ell_N$ depends strongly on N. From the figure it is evident that Δx_1^1, the width of the transparency region for the mode $N = 1$, is quite appreciable in the magnetosphere, from 10^3 to 10^4 km. This, projected into the ionosphere, gives some 100 km. But values of Δx_N^1 when $N > 1$ are very small, 100 km or less.

The plot of k_{1N}^2 vs the coordinate x^1 for the mode $N = 1$ in Fig. 5 totally agrees with all qualitative conclusions of the preceding section. With our choice of the coordinates $x^1 = a$ and $x^2 = \varphi$, the physical components of the wavevector $\hat{k}_1 = k_1/\sqrt{g_1}$, $\hat{k}_2 = k_2/\sqrt{g_2} = m/\sqrt{g_2}$ on the equator are $\hat{k}_1 = k_1$, $\hat{k}_2 = m/a$. Thus, in Fig. 5 the abscissa axis indicates the square of the ratio of equatorial values of physical components of the wavevector $\hat{k}_1^2/\hat{k}_2^2 = k_1^2a^2/m^2$. The asymptotic value of this ratio for $x^1 - x_{TN}^1 \to \infty$ is $a^2/p_{\min}^2 = -1$, and for $x^1 - x_{PN}^1 \to -\infty$, it is $a^2/p_{\max}^2 = -(1+3\sin^2\theta_*) = -(4-3R_1/a)^{-1}$. By analysing the definitions (28) and (29) of the quantities w_N^{P} and w_N^{T}, it becomes possible to obtain the following estimates:

$$w_N^{\mathrm{P}} \sim \alpha_N A_0^2, \qquad w_N^{\mathrm{T}} \sim \alpha_N A_0^2/a^4, \quad (53)$$

where A_0 is a characteristic value of Alfvén velocity. Similarly, an estimate of characteristic equatorial values of physical components of group velocity $\hat{v}_N^1 = v_N^1$ and $\hat{v}_N^2 = av_N^2$:

$$\hat{v}_N^1 \sim \hat{v}_N^2 \sim \alpha_N A/m$$

follows from equations (33a,b). It agrees well with the plots given in Fig. 6.

7. The damping of longitudinal modes on the ionosphere

In order to take account of the wave energy dissipation in the ionosphere, it is necessary to retain the first term on the right-hand side of the boundary condition (12). Instead of equations (15) and (16), we then arrive at the following problem for eigenvalues:

$$\hat{L}(x^1, k_1, k_2, \omega)H = 0, \qquad H|_{\ell_\pm} = \mp\mathrm{i}(v_\pm/\omega)(\partial H/\partial\ell)|_{\ell_\pm}. \quad (54)$$

Its solutions will differ from former eigenvalues of k_{1N} for some corrections δk_{1N}:

$$k_1 = k_{1N}(x^1, \omega) + \delta k_{1N}(x^1, \omega).$$

As is known, the quantity δk_{1N} is related to a local damping decrement of the mode $\gamma_N = \gamma_N(x^1, \omega)$ by the following relationship:

$$\delta k_{1N} = \mathrm{i}\gamma_N/v_N^1. \quad (55)$$

Indeed, when it is satisfied, the correction to the quasi-classical phase can be put into the form:

$$\delta\tilde{\Psi} = \int_{x_{PN}^1}^{x^1} \delta k_{1N}\,\mathrm{d}x^{1'} \equiv \mathrm{i}\Gamma(x^1), \quad (56)$$

where we introduce the designation:

$$\Gamma(x^1) = \int_{x_{PN}^1}^{x^1} \frac{\gamma_N}{v_N^1}\,\mathrm{d}x^{1'} = \int_0^\tau \gamma_N\,\mathrm{d}\tau'.$$

The definition (39) is used in the last equality. Thus, in accordance with equation (14), we have the factor:

$$\exp(\mathrm{i}\delta\tilde{\Psi}) = \exp(-\Gamma),$$

which describes the attenuation of the wave as it propagates along the characteristic. Note that if this matter is treated strictly formally, then the equality (55) should be considered as the definition of the decrement γ_N.

For the actual determination of the correction δk_{1N}, we use perturbation theory. Put $H = f_N(R_N + h_N)$, where f_N is a constant that does not depend on ℓ, and we linearize the problem (54) in small corrections δk_{1N} and h_N:

$$\hat{L}(x^1, k_{1N}, k_2, \omega)h_N + \delta k_{1N} \frac{\partial \hat{L}(x^1, k_{1N}, k_2, \omega)}{\partial k_1} R_N = 0;$$

$$h_N|_{\ell_\pm} = \mp i \frac{v_\pm}{\omega} \frac{\partial R_N}{\partial \ell}\bigg|_{\ell_\pm}. \quad (57)$$

We multiply the first of these relationships by R_N and integrate along the field line. As a result, we obtain:

$$\delta k_{1N} = -\oint R_N \hat{L} h_N \, d\ell \bigg/ \oint R_N \frac{\partial \hat{L}}{\partial k_1} R_N \, d\ell. \quad (58)$$

Since $\partial \hat{L}/\partial k_1 = 2k_1 \hat{L}_T$, then in accordance with equation (33a):

$$\oint R_N \frac{\partial \hat{L}}{\partial k_1} R_N \, d\ell = 2k_1 \oint R_N \hat{L}_T R_N \, d\ell = -2\omega v_N^1.$$

By then comparing equations (55) and (58), we obtain the equality:

$$\oint R_N \hat{L} h_N \, d\ell = 2i\omega\gamma_N. \quad (59)$$

On transforming its left-hand side by means of integration by parts and using the boundary condition, we obtain the following expression for the decrement:

$$\gamma_N = \frac{1}{\omega^2}\left[q_N^{(+)} v_+ \left(\frac{\partial R_N}{\partial \ell}\right)^2_{\ell_+} + q_N^{(-)} v_- \left(\frac{\partial R_N}{\partial \ell}\right)^2_{\ell_-}\right]. \quad (60)$$

Here it is designated $q_N^{(\pm)} = q_N(\ell_\pm) = p_\pm k_{1N}^2 + p_\pm^{-1} k_2^2$, where $p_\pm = p(\ell_\pm)$. Together with the equality (55), this formula defines the correction δk_{1N}. Note also that the local decrement γ_N can be considered as a correction to the eigenfrequency [equation (23)] caused by the damping on the ionosphere: $\omega = \omega_N - i\gamma_N$. In this case the decrement should be regarded as a function of the variables x^1 and κ: $\gamma_N = \gamma_N[x^1, \omega_N(x^1, \kappa)]$. Near the poloidal surface, from equation (60) we have:

$$\gamma_N = \frac{1}{\omega^2}\left[\frac{v_+}{p_+}\left(\frac{\partial P_N}{\partial \ell}\right)^2_{\ell_+} + \frac{v_-}{p_-}\left(\frac{\partial P_N}{\partial \ell}\right)^2_{\ell_-}\right] \quad (61)$$

and near the toroidal surface:

$$\gamma_N = \frac{1}{\omega^2}\left[p_+ v_+ \left(\frac{\partial T_N}{\partial \ell}\right)^2_{\ell_+} + p_- v_- \left(\frac{\partial T_N}{\partial \ell}\right)^2_{\ell_-}\right]. \quad (62)$$

For large numbers N, when formulas of the longitudinal WKB approximation are applicable, on substituting the expression (44) into equation (60), we get:

$$\gamma_N = \frac{2}{t_A}\left(\frac{v_+}{A_+} + \frac{v_-}{A_-}\right), \quad (63)$$

where $A_\pm = A(\ell_\pm)$ is the value of Alfvén velocity on the boundary with the ionosphere. In this approximation γ_N does actually not depend on N. Formula (63) can be used to estimate γ_N. The typical values of $A_\pm \sim (3\times10^3$–$3\times10^4)$ km s^{-1}, $v_\pm \sim 10^2$ km s^{-1} for the dayside ionosphere, and $v_\pm \sim 10^3$ km s^{-1} for the nightside ionosphere. Then $v_\pm/A_\pm \sim 0.03$–0.003 for the dayside ionosphere and $v_\pm/A_\pm \sim 0.3$–0.03 for the nightside ionosphere. The smallness of the parameters $v_\pm/A_\pm$ ensures the weakness of the damping of the mode, i.e. the smallness of the decrement γ_N as compared with the difference of neighbouring eigenfrequencies (say, ω_N and ω_{N+1}) which, on the order of magnitude, is $1/t_A$.

8. Distribution of the oscillation amplitude in transverse coordinate

The main order of the WKB approximation in coordinate x^1 equations (15) and (16) defines the function $H = H_N$ up to an arbitrary factor which can depend on x^1 and ω. In other words, the solution in this order can be represented as:

$$H_N(x^1, \ell, \omega) = f_N(x^1, \omega) R_N(x^1, \ell, \omega). \quad (64)$$

The function f_N should be treated as the standing wave amplitude on a given magnetic shell. The equation defining f_N is the solvability condition for the correction of the next order of WKB approximation in coordinate x^1.

Put:

$$H = H_N + \tilde{h}_N = f_N(R_N + h_N). \quad (65)$$

Here, the correction of the next order $\tilde{h}_N$ is represented as $\tilde{h}_N = f_N h_N$, which does not detract from generality. It will be assumed that the function h_N involves not only the corrections associated with the next order of WKB approximation, but also the damping in the ionosphere. This means that the function h_N satisfies the boundary condition (57).

On substituting equation (65) into equation (4), in the next (after the main) order of WKB approximation we have:

$$-f_N \hat{L} h_N + i[\nabla_1(k_1 f_N \hat{L}_T R_N) + k_1 \hat{L}_T \nabla_1(f_N R_N)] = 0.$$

We multiply this equation by $f_N R_N$ and integrate along the field line. Taking into account the Hermitian character of the operator $\hat{L}_T$ with respect to the functions that go to zero when $\ell = \ell_\pm$, we arrive at the relationship:

$$\nabla_1 k_1 f_N^2 \oint R_N \hat{L}_T R_N \, d\ell + i f_N^2 \oint R_N \hat{L}_T h_N \, d\ell = 0.$$

Finally, using the equalities (33) and (59), we obtain:

$$\nabla_1 v_N^1 f_N^2 = -2\gamma_N f_N^2. \quad (66)$$

The relationship obtained has a simple physical meaning. Let ε be the energy density of Alfvén oscillations. It consists of the energy of the disturbed magnetic field and the kinetic energy of the plasma particles:

$$\varepsilon = \frac{B^2}{8\pi} + \frac{\rho v_E^2}{2},$$

where $\mathbf{v}_E = c[\mathbf{E}\mathbf{B}_0]/B_0^2$ is the electric drift velocity of plasma in the wave's field. On substituting here the relationships (21) and (64), we obtain:

$$\varepsilon = \frac{1}{8\pi}\left(\frac{B_1B_1^*}{g_1} + \frac{B_2B_2^*}{g_2}\right) + \frac{\rho c^2}{2B_0^2}\left(\frac{E_1E_1^*}{g_1} + \frac{E_2E_2^*}{g_2}\right)$$
$$= \frac{c^2}{8\pi\omega^2}\frac{f_N^2}{\sqrt{g_\perp}}\left[q_N\left(\frac{\partial R_N}{\partial \ell}\right)^2 + q_N\frac{\omega^2}{A^2}R_N^2\right].$$

We then calculate the value of $\bar{\varepsilon}$, the energy of the oscillation contained by a thin flux tube with unit size in coordinates x^1 and x^2 (i.e. $dx^1 = 1$ and $dx^2 = 1$). The cross-sectional area of such a tube $\sigma = \sqrt{g_1g_2} = \sqrt{g_\perp}$. We have:

$$\bar{\varepsilon} = \frac{1}{2}\oint \varepsilon\sigma\, d\ell$$
$$= \frac{c^2}{16\pi}f_N^2\oint\left[\frac{q_N}{\omega^2}\left(\frac{\partial R_N}{\partial \ell}\right)^2 + \frac{q_N}{A^2}R_N^2\right]d\ell = \frac{c^2}{8\pi}f_N^2.$$

On the other hand, by integrating the transverse contravariant components of the Poynting vector:

$$S^1 = \frac{c}{4\pi}\frac{1}{\sqrt{g}}E_2^*B_3, \qquad S^2 = -\frac{c}{4\pi}\frac{1}{\sqrt{g}}E_1^*B_3,$$

over the volume of the same flux tube, we find the contravariant vector components of the tube's transverse energy flux:

$$\bar{S}^i = \frac{1}{2}\oint S^i\sigma\, d\ell = \frac{c^2}{8\pi}v_N^i f_N^2 = v_N^i\bar{\varepsilon}. \qquad (67)$$

The relationships (21), (64) and (33) are used here. From the formulas obtained, it is evident that the equality (66) is the equation of oscillation energy balance of the flux tube

$$\nabla_i\bar{S}^i = -2\gamma_N\bar{\varepsilon}.$$

In this case it is, of course, taken into consideration that, by virtue of axial symmetry, $\nabla_2\bar{S}^2 = 0$. It is easy to solve equation (66):

$$f_N = Cv_N^{-1/2}\exp(-\Gamma), \qquad (68)$$

where C is a constant that does not depend on x^1. It should be noted that the exponential part of this formula was essentially determined in the preceding section [formula (56)]. By combining formulas (14), (20), (64) and (68), we obtain the solution in two main orders of WKB approximation:

$$\Phi(x^1,\ell,\omega) = \frac{C}{\sqrt{v_N^1}}$$
$$\times \exp\left(i\int_{x_{PN}^1}^{x^1} k_{1N}\,dx^{1'} - \int_{x_{PN}^1}^{x^1}\frac{\gamma_N}{v_N^1}dx^{1'}\right)R_N(x^1,\ell,\omega). \quad (69)$$

From the relationship (36) it is evident that the integral in the exponent of formula (68) converges on the lower limit. Consequently, the amplitude f_N itself, due to the presence of a pre-exponent, has the singularity $f_N \sim (x^1 - x_{PN}^1)^{-1/4}$ on the poloidal surface. With distance from the poloidal surface, f_N decreases both due to an increase in group velocity v_N^1 and as a consequence of the damping on the ionosphere. On the toroidal surface the pre-exponent in equation (68) becomes infinite by the law $(x_{TN}^1 - x^1)^{-3/4}$. But since the integral in the exponent also tends to infinity (due to the infinite time taken by the wave to approach the toroidal surface), the full amplitude f_N tends to zero.

The behaviour of the amplitude will be different in the presence of an instability of the Alfvén waves considered here [for a review of possible instabilities see, for example, Southwood and Hughes (1983)]. If the action of the instability mechanism is stronger than the dissipation in the ionosphere, then the wave will be excited rather than damped. Such a situation can be described phenomenologically if the quantity γ_N is considered negative. Then, the exponential representation in equation (69) increases from the poloidal to the toroidal surface, and on the latter it becomes infinite.

To conclude this section, let us discuss the crucial question of the applicability conditions of the WKB approximation in coordinate x^1. The initial form of this condition is known to be:

$$\left|\frac{d}{dx^1}\frac{1}{k_1}\right| \ll 1. \qquad (70)$$

For analysing this inequality, we use the model expression

$$\hat{k}_1^{0^2} = \hat{k}_2^{0^2}\frac{x^1 - x_{PN}^1}{x_{TN}^1 - x^1}. \qquad (71)$$

Here $\hat{k}_{1,2}^0 = k_{1,2}/\sqrt{g_{1,2}(0)}$ are the physical components of the wavevector in the equatorial plane $\ell = 0$. Not only does the expression (71) represent correctly the qualitative behaviour features of the function $\hat{k}_1^2(x^1)$, but it is also true on the order of magnitude.

From the expression (71) it follows that the inequality (70) can be satisfied only if:

$$\hat{k}_2\Delta\hat{x}_N \gg 1, \qquad (72)$$

where $\Delta\hat{x}_N = \sqrt{g_1}(x_{TN}^1 - x_{PN}^1)$ is a typical physical distance between the resonance surfaces (in the equatorial plane, say). Since between the resonance surfaces, on the order of magnitude, $\hat{k}_1 \sim \hat{k}_2$, then the condition (72) means that many transverse wavelengths find room between these surfaces. Since on the order of magnitude $\hat{k}_2^0 = m/a$, the inequality (72) can be rewritten as $m \gg a/\Delta\hat{x}_N \sim 1/\alpha_N$. For the fundamental harmonic ($N = 1$), this leads to the condition which does actually not differ from our originally assumed $m \gg 1$. However, already for the next harmonic ($N = 2$) the requirement for the azimuthal wavenumber becomes more stringent: $m \gg 10$–10^2, depending on the geomagnetic field model.

Even if the inequality (72) is satisfied, the condition (70) is violated near the resonance surfaces. Assuming that inequality (72) is satisfied, it is easy to see that the WKB approximation is applicable if:

$$|x^1 - x_{PN}^1| \gg \lambda_{PN}, \qquad |x^1 - x_{TN}^1| \gg \lambda_{TN}, \qquad (73)$$

where:

$$\lambda_{PN} \sim \Delta x_N^1/(\hat{k}_2\Delta\hat{x}_N)^{2/3}, \qquad \lambda_{TN} \sim \Delta x_N^1/(\hat{k}_2\Delta\hat{x}_N)^2. \quad (74)$$

It should be emphasized that $\lambda_{PN}, \lambda_{TN} \ll \Delta x_N^1$. Thus, in the neighbourhoods of the resonance surfaces $|x^1 - x_{PN}^1| \lesssim \lambda_{PN}$ and $|x^1 - x_{TN}^1| \lesssim \lambda_{TN}$ the formulas of transverse WKB approximation are inapplicable. In particular, the conclusions that the amplitude f_N on the resonance surfaces becomes infinite (or goes to zero), are wrong. For investigating the solution in these neighbourhoods, it is necessary to abandon, by returning to equation (4), the WKB approximation.

9. Solution near the poloidal resonance surface

For disturbances that depend on the coordinate x^2 by the law $\exp(ik_2x^2)$, equation (4) assumes the form:

$$[\nabla_1 \hat{L}_T(\omega)\nabla_1 - k_2^2 \hat{L}_P(\omega)]\Phi = 0. \quad (75)$$

We shall find its solution near the poloidal surface using perturbation theory based on the fact that the desired solution is close to the poloidal mode. This implies:

$$|\nabla_1/\sqrt{g_1}| \ll k_2/\sqrt{g_2}, \quad (76)$$

i.e. the first term in equation (75) is a small one. In the main order of perturbation theory, by omitting this term and using also the zeroth-order approximation for the boundary condition, we have:

$$\hat{L}_P(\omega)\Phi = 0, \qquad \Phi|_{\ell_\pm} = 0.$$

The solution of this problem for eigenvalues is well known to us:

$$\Phi = U_N P_N, \qquad \omega^2 = \Omega_N^{P^2}, \quad (77)$$

where U_N is a factor that does not depend on ℓ. By comparing equation (77) with equations (14) and (64), one can see that in the region where the WKB approximation in transverse coordinate is applicable:

$$U_N = (1/k_2) f_N \exp(i\Psi). \quad (78)$$

In the main order of the perturbation theory developed here, the factor U_N is not defined because the solution (77) is degenerate. The equation for U_N, as usual in such a case, represents the solvability condition for the correction of the next approximation.

We put:

$$\Phi = U_N(x^1, \omega)P_N(x^1, \ell) + \varphi_N$$

and linearize equation (75), letting the first term in this equation, the function φ_N and the difference $\omega^2 - \Omega_N^{P^2}$ be small quantities. By multiplying the obtained equality by P_N and integrating along the field line, we obtain:

$$\nabla_1^2 U_N \oint P_N \hat{L}_T(\Omega_N^P) P_N \, d\ell - k_2^2(\omega^2 - \Omega_N^{P^2})U_N - k_2^2 \oint P_N \hat{L}_P(\Omega_N^P)\varphi_N \, d\ell = 0. \quad (79)$$

For calculating the last term in this relationship, we linearize the boundary condition (12):

$$\Phi_N|_{\ell_\pm} = \mp i \frac{v_\pm}{\omega} \frac{\partial P_N}{\partial \ell}\bigg|_{\ell_\pm} U_N - \frac{J^{(\pm)}}{V_\pm}. \quad (80)$$

Here we have, finally, included the term with extraneous currents in the ionosphere. It will be demonstrated below that it should, indeed, be taken into account only in the immediate vicinity of the poloidal surface. By means of integration by parts, using the relationship (61), we obtain:

$$\oint P_N \hat{L}_P(\Omega_N^P)\Phi_N \, d\ell = 2i\omega\gamma_N U_N + I_\parallel,$$

where it is designated:

$$I_\parallel = 2\left[-\left(\frac{\partial P_N}{\partial \ell}\right)_{\ell_+} \frac{J_\parallel^{(+)}}{p_+ V_+} + \left(\frac{\partial P_N}{\partial \ell}\right)_{\ell_-} \frac{J_\parallel^{(-)}}{p_- V_-}\right]. \quad (81)$$

Taking also the equality (28) into account, from equation (79) we obtain the desired equation for the function U_N:

$$w_N^P \nabla_1^2 U_N + k_2^2[(\omega + i\gamma_N)^2 - \Omega_N^{P^2}]U_N = k_2^2 I_\parallel. \quad (82)$$

We wish to stress that this is an inhomogeneous equation and defines the solution together with its amplitude.

The coefficients of equation (82) are functions of coordinate x^1. The most important dependence on x^1 belongs to the coefficient of U_N enclosed in square brackets. On the poloidal surface it nearly goes to zero. The other quantities, w_N^P, $I_\parallel$ and γ_N, can usually be considered constant in the region of our interest. If the poloidal surface lies not too close to the extremum of the function $\Omega_N^P(x^1)$, the expansion (26) then holds near this surface. Then:

$$(\omega + i\gamma_N)^2 - \Omega_N^{P^2} = \omega^2\left(\frac{x^1 - x_{PN}^1}{\ell_N} + 2i\frac{\gamma_N}{\omega}\right).$$

We introduce the dimensionless variable:

$$\xi = (x^1 - x_{PN}^1)/\lambda_{PN},$$

be defining the constant λ_{PN} by the relationship:

$$\lambda_{PN} = \left(\frac{w_N^P \ell_N}{k_2^2 \omega^2}\right)^{1/3}. \quad (83)$$

Equation (82) can then be brought into the form:

$$\frac{d^2 U_N}{d\xi^2} + (\xi + i\varepsilon_{PN})U_N = \frac{\ell_N}{\lambda_{PN}} \frac{I_\parallel}{\omega^2}, \quad (84)$$

where it is designated:

$$\varepsilon_{PN} = 2\frac{\ell_N}{\lambda_{PN}} \frac{\gamma_N}{\omega}.$$

It is easy to see that the quantity λ_{PN} has the same dimension as the coordinate x^1 and is the typical scale of the solution in this coordinate (for $\varepsilon_{PN} \lesssim 1$). The parameter ε_{PN} characterizes the role of the damping on the ionosphere. When $\varepsilon_{PN} \ll 1$ this role is negligible, and when $\varepsilon_{PN} \gg 1$, on the contrary, it is the damping that determines the form of the solution, namely the first term in equation (82) in this case can be omitted. For the magnetospheric model described in Section 6, in view of the relationship (53), we have the estimation:

$$\lambda_{PN} \sim \alpha_N^{1/3} a/m^{2/3}. \quad (85)$$

It wholly agrees with the definition (74). From this estimation it follows that the condition for the poloidal character of the mode [equation (76)] equivalent to the inequality $\bar{k}_2\bar{\lambda}_{PN} \gg 1$, is satisfied if $m \gg 1/\alpha_N$. But we assume the last inequality to be satisfied.

For the unique solution of equation (84), it is necessary to impose the boundary conditions in coordinate ξ. In the opacity region, $\xi \to -\infty$, the boundedness condition for the solution seems natural. In the transparency region, $\xi \to \infty$, we require that the solution have the form of an escaping wave that carries along the energy from the poloidal surface. Such a requirement is based upon the remarkable property of the solution near the toroidal surface. In the next section, it will be shown that the wave incident on the toroidal surface is totally absorbed on it. This means that there is no wave reflected from the toroidal surface and travelling toward the poloidal surface.

The desired solution of equation (84) is expressed in terms of a standard function $G(z)$ which satisfies the inhomogeneous Airy equation:

$$G''+zG = 1 \tag{86}$$

and the above boundary conditions. This function has the following integral representation

$$G(z) = -\mathrm{i}\int_0^{\infty} \exp(\mathrm{i}sz - \mathrm{i}s^3/3)\,\mathrm{d}s,$$

and the asymptotic representations:

$$G(z) = \begin{cases} \dfrac{1}{z}, & z \to -\infty; \quad (87a) \\ -\dfrac{\pi^{1/2}}{z^{1/4}}\exp\left(\dfrac{2}{3}\mathrm{i}z^{3/2}+\mathrm{i}\dfrac{\pi}{4}\right), & z \to \infty; \quad (87b) \end{cases}$$

from which it is evident that it does, indeed, satisfy the requirement conditions. The function $G(z)$ was described in greater detail by Leonovich and Mazur (1989a) [they denoted it by $\phi(z)$]. An important property of the function $G(z)$ becomes clear when comparing it in the asymptotic region $z \to \infty$ with the solution of the homogeneous Airy equation:

$$G''+zG = 0.$$

By applying, when $z \gg 1$, the WKB approximation to the last equation, we obtain a general solution for it:

$$G(z) = \frac{A}{z^{1/4}}\exp\left(\frac{2}{3}\mathrm{i}z^{3/2}\right)+\frac{B}{z^{1/4}}\exp\left(-\frac{2}{3}\mathrm{i}z^{3/2}\right).$$

Comparing with equation (87b) one can see that the function $G(z)$ in the asymptotic region coincides with one of the solutions of the homogeneous equation. In other words, the presence of the right-hand side in equation (86) is substantial only for $z \sim 1$, and in the region $z \gg 1$ the right-hand side can be omitted because this will not affect the solution. In this case, of course, one should bear in mind that the amplitude and phase of the solution in the asymptotic region are defined by the right-hand side in the region $z \sim 1$. An investigation by means of the Green's function shows that such a property of the function $G(z)$ is due to the reduction in size of the spatial scale of the solution when moving farther into the region $z \gg 1$. The smaller the scale of the solution is, the greater is the reason for neglecting the inhomogeneous term of the equation. When applied to the solution of equation (84) given below, the property of the function $G(z)$ concerned means that the inhomogeneous term in the boundary condition (80) should be taken into account only in the region $|x^1 - x^1_{PN}| \sim \lambda_{PN}$. This justifies ignoring it when using the WKB approximation in coordinate x^1.

Using the function $G(z)$, the solution of equation (84) is representable as:

$$U_N = \frac{\ell_N I_\parallel}{\lambda_{PN}\omega^2}G(\xi+\mathrm{i}\varepsilon_{PN}) = \frac{\ell_N I_\parallel}{\lambda_{PN}\omega^2}G\left(\frac{x^1-x^1_{PN}}{\lambda_{PN}}+\mathrm{i}\varepsilon_{PN}\right). \tag{88}$$

In accord with formula (77), for the disturbed electric field potential we have:

$$\Phi = \frac{\ell_N I_\parallel}{\lambda_{PN}\omega^2}G\left(\frac{x^1-x^1_{PN}}{\lambda_{PN}}+\mathrm{i}\varepsilon_{PN}\right)P_N(x^1,\ell). \tag{89}$$

Remember that these formulas are applicable near the poloidal surface when $|x^1 - x^1_{PN}| \ll \Delta x^1_N$.

10. The solution near the toroidal resonance surface

As in the preceding section, we shall seek the solution using perturbation theory, under the assumption that it is close to the toroidal mode. This means that the inequality:

$$|\nabla_1/\sqrt{g_1}| \gg k_2/\sqrt{g_2}, \tag{90}$$

opposite to equation (76), is valid, and the second term in square brackets of equation (75) should be considered small. In the main order of perturbation theory, we have:

$$\Phi = V_N T_N, \qquad \omega^2 = \Omega_N^{T^2}, \tag{91}$$

where V_N is a factor independent of ℓ. In the region where the WKB approximation in coordinate x^1 is applicable, we have:

$$V_N = (1/k_1)f_N\exp(\mathrm{i}\psi). \tag{92}$$

In the next order, we assume:

$$\Phi = V_N(x^1,\omega)T_N(x^1,\ell)+\psi_N.$$

The linearization of the boundary condition yields:

$$\psi_N|_{\ell_\pm} = \mp\mathrm{i}\frac{v_\pm}{\omega}\left.\frac{T_N}{\partial\ell}\right|_{\ell_\pm}V_N. \tag{93}$$

Here the term with extraneous currents is omitted because of the extremely small-scale character of the solution (see below). We linearize equation (75), multiply the relationship obtained by T_N, and integrate along the field line. As a result, we get:

$$\oint T_N \hat{L}_T(\Omega_N^T)\nabla_1^2\psi_N \, d\ell + \nabla_1(\omega^2-\Omega_N^{T^2})\nabla_1 V_N - k_2^2 V_N \oint T_N \hat{L}_P(\Omega_N^T) T_N \, d\ell = 0.$$

Integrating by parts using the relationship (93) and using formula (62) we have:

$$\oint T_N \hat{L}_T(\Omega_N^T)\nabla_1^2\psi_N \, d\ell = 2i\omega\gamma_N \nabla_1^2 V_N.$$

Taking also the definition (29) into account we obtain the desired equation for V_N:

$$\nabla_1[(\omega+i\gamma_N)^2-\Omega_N^{T^2}]\nabla_1 V_N - k_2^2 w_N^T V_N = 0. \quad (94)$$

Let us find the solution of this equation for the case when the expansion (26) holds for the function $\Omega_N^T(x^1)$. Introduce the dimensionless variable:

$$\eta = (x^1 - x_{TN}^1)/\lambda_{TN},$$

by defining the constant λ_{TN} by the relationship

$$\lambda_{TN} = \frac{\omega^2}{k_2^2 w_N^T \ell_N}. \quad (95)$$

Equation (94) can then be rewritten in the form:

$$\frac{d}{d\eta}(\eta + i\varepsilon_{TN})\frac{dV_N}{d\eta} - V_N = 0, \quad (96)$$

where:

$$\varepsilon_{TN} = 2\frac{\ell_N \gamma_N}{\lambda_{TN}\omega}.$$

As in the preceding section, the quantity λ_{TN} stands for the typical scale of the solution in the neighbourhood of the toroidal surface, and the parameter ε_{TN} characterizes the role of the ionospheric damping in this neighbourhood. For the magnetospheric model used here, in view of the estimation (53), on the order of magnitude we have:

$$\lambda_{TN} \sim a/(\alpha_N m^2). \quad (97)$$

This relationship is also consistent with the definition (74). The toroidality condition (90), in view of equation (97), again leads to the inequality $m \gg 1/\alpha_N$.

Introduce into our treatment the function $g(z)$ that satisfies the equation:

$$(zg')' - g = 0,$$

and is bounded when $z \to \infty$. This function is expressed in terms of one of the cylindrical functions, the modified zeroth-order Hankel function:

$$g(z) = K_0(2z^{1/2}). \quad (98)$$

It has, as $z \gg 1$, the following asymptotic representation

$$g(z) \approx (\sqrt{\pi}/2)z^{-1/4}\exp(-2z^{1/2}). \quad (99)$$

At small z:

$$g(z) \approx -(1/2)\ln z. \quad (100)$$

The point $z = 0$ is a singular (branch) point. The behaviour of the function $g(z)$ at negative z is determined by the way in which the singular point is indented. In our case this method is due to the presence of the damping, and is specified by the rule $z = \eta + i\varepsilon_{TN}$. This leads to the following asymptotic representation when $z \to -\infty$:

$$g(z) \approx (\sqrt{\pi}/2)(-z)^{-1/4}\exp[-2i(-z)^{1/2} - i\pi/4]. \quad (101)$$

The solution of equation (96) has the form:

$$V_N = Dg(\xi + i\varepsilon_{TN}) = Dg\left(\frac{x^1 - x_{TN}^1}{\lambda_{TN}} + i\varepsilon_{TN}\right), \quad (102)$$

where D is an arbitrary constant. In accordance with equation (91), from this we have:

$$\Phi = Dg\left(\frac{x^1 - x_{TN}^1}{\lambda_{TN}} + i\varepsilon_{TN}\right)T_N(x^1, \ell). \quad (103)$$

This solution, in the transparency region, represents a wave escaping toward the toroidal surface and absorbed in its neighbourhood on a scale $|x^1 - x_{TN}^1| \sim \lambda_{TN}$. The reflected wave is totally absent. In the opacity region the solution decreases exponentially with the distance from the toroidal surface. It should be stressed that formulas (102) and (103) hold when $|x^1 - x_{TN}^1| \ll \Delta x_N^1$.

11. Global structure of the mode (matching of solutions in different regions)

In order to obtain a full description of the spatial structure of the mode, it is necessary to match the solutions obtained in preceding sections for different regions in x^1. With such a description, it will be convenient for us to use the dimensionless function $r_N(x^1, \ell)$ defined by the equality:

$$r_N = (q_N t_A/A)^{1/2} R_N.$$

It satisfies the relationship:

$$\langle r_N^2 \rangle = 1,$$

where, for the arbitrary function $F = F(\ell)$, it is designated:

$$\langle F \rangle = \frac{1}{t_A}\oint F(\ell)\frac{d\ell}{A},$$

being an average along the field line. Near the poloidal and toroidal surfaces, we have, respectively:

$$r_N = (t_A/pA)^{1/2} P_N, \qquad r_N = (t_A p/A)^{1/2} T_N.$$

For large N when the WKB approximation in coordinate ℓ is applicable:

$$r_N = \sqrt{2}\sin\left(\frac{2\pi N}{t_A}\int_{\ell}^{\ell'} \frac{d\ell'}{A}\right).$$

Using this definition the solution (89) for a perturbed potential near the poloidal surface will be represented as:

$$\Phi = \tilde{\Phi}\left(\frac{pA}{p_0A_0}\right)^{1/2} G\left(\frac{x^1 - x^1_{PN}}{\lambda_{PN}} + i\varepsilon_{PN}\right) r_N(x^1, \ell), \quad (104)$$

where:

$$\tilde{\Phi} = \left(\frac{p_0A_0}{t_A}\right)^{1/2} \frac{\ell_N}{\lambda_{PN}} \frac{I_\parallel}{\omega^2},$$

is a typical value of perturbed potential. Here, as in the above, the index zero denotes the equatorial values of corresponding quantities. The validity ranges of the solution (104) and of the WKB approximation in transverse coordinate overlap. In the overlapping region $\lambda_{PN} \ll x^1 - x^1_{PN} \ll \Delta x^1_N$, in accordance with equation (87b), from equation (104) we have:

$$\Phi = -\sqrt{\pi}\tilde{\Phi}\left(\frac{pA}{p_0A_0}\right)^{1/2}\left(\frac{\lambda_{PN}}{x^1 - x^1_{PN}}\right)^{1/4}$$

$$\times \exp\left[\frac{2}{3}i\left(\frac{x^1 - x^1_{PN}}{\lambda_{PN}}\right)^{3/2} - \varepsilon_{PN}\left(\frac{x^1 - x^1_{PN}}{\lambda_{PN}}\right)^{1/2} + i\frac{\pi}{4}\right] r_N. \quad (105)$$

On the other hand, from equations (30) and (34), in view of the definition (83), we have:

$$k_{1N} = \frac{(x^1 - x^1_{PN})^{1/2}}{\lambda_{PN}^{3/2}}, \qquad v^1_N = v^1_{PN}\left(\frac{x^1 - x^1_{PN}}{\lambda_{PN}}\right)^{1/2},$$

where it is designated $v^1_{PN} = \omega\lambda^2_{PN}/\ell_N$ is a typical value of transverse group velocity in the neighbourhood of the poloidal surface, from which:

$$\psi = \int_{x^1_{PN}}^{x^1} k_{1N}\,dx^{1'} = \frac{2}{3}\left(\frac{x^1 - x^1_{PN}}{\lambda_{PN}}\right)^{3/2},$$

$$\Gamma = \int_{x^1_{PN}}^{x^1} \frac{\gamma_N}{v^1_N}\,dx^{1'} = \varepsilon_{PN}\left(\frac{x^1 - x^1_{PN}}{\lambda_{PN}}\right)^{1/2}.$$

By comparing equation (69) with equation (105) in the overlapping region, one can see that functionally they wholly coincide, in both the coordinate x^1 and the coordinate ℓ. This permits us to define the constant C. Substituting it into the general formula (69), we obtain the solution in the validity range of the WKB approximation matched with the solution in the neighbourhood of the poloidal surface:

$$\Phi = -\sqrt{\pi}\tilde{\Phi}\left(\frac{v^1_{PN}}{v^1_N}\frac{p^{-1}k_2^2}{pk^2_{1N} + p^{-1}k_2^2}\frac{pA}{p_0A_0}\right)^{1/2}$$

$$\times \exp\left(i\int_{x^1_{PN}}^{x^1} k_{1N}\,dx^{1'} - \int_{x^1_{PN}}^{x^1}\frac{\gamma_N}{v^1_N}\,dx^{1'} + i\frac{\pi}{4}\right) r_N. \quad (106)$$

In much the same way, we match the solution (106) with the solution in the neighbourhood of the toroidal surface. In their common validity range $\lambda_{TN} \ll x^1_{TN} - x^1 \ll \Delta x^1_N$, from equations (30) and (35), in view of the definition (95), we have:

$$k_{1N} = \frac{1}{\lambda^{1/2}_{TN}(x^1_{TN} - x^1)^{1/2}}, \qquad v^1_N = v^1_{TN}\left(\frac{x^1_{TN} - x^1}{\lambda_{TN}}\right)^{3/2},$$

where $v^1_{TN} = \omega\lambda^2_{TN}/\ell_N$ is a typical value of v^1_N in the neighbourhood of the toroidal surface. We put:

$$\bar{\psi} = \int_{x^1_{PN}}^{x^1_{TN}} k_{1N}\,dx^1, \quad (107)$$

being a full run-on of the quasiclassical phase between the resonance surfaces. It must be emphasized that the integral (107) converges on the upper limit, on the order of magnitude $\bar{\psi} \sim \bar{k}_2\Delta x^1_N \sim m\alpha_N \gg 1$. In view of this definition, for $0 \leqslant x^1_{TN} - x^1 \ll \Delta x^1_N$ we have:

$$\psi(x^1) = \bar{\psi} - \int_{x^1}^{x^1_{TN}} k_{1N}\,dx^{1'} = \bar{\psi} - 2\left(\frac{x^1_{TN} - x^1}{\lambda_{TN}}\right)^{1/2}.$$

Unfortunately, proceeding in this way it is impossible to calculate the value of $\Gamma(x^1)$ because the integral defining it diverges when $x^1 \to x^1_{TN}$. Introduce an auxiliary coordinate $\bar{x}^1$ enclosed in the same limits: $0 < x^1_{TN} - \bar{x}^1 \ll \Delta x^1_N$. One can then write:

$$\Gamma(x^1) = \int_{x^1_{PN}}^{\bar{x}^1}\frac{\gamma_N}{v^1_N}\,dx^{1'} + \int_{\bar{x}^1}^{x^1}\frac{\gamma_N}{v^1_N}\,dx^{1'}$$

$$= \Gamma(\bar{x}^1) - \varepsilon_{TN}\left(\frac{\lambda_{TN}}{x^1_{TN} - \bar{x}^1}\right)^{1/2} + \varepsilon_{TN}\left(\frac{\lambda_{TN}}{x^1_{TN} - x^1}\right)^{1/2}.$$

It is easy to make sure that the quantity:

$$\bar{\Gamma} = \Gamma(\bar{x}^1) - \varepsilon_{TN}\left(\frac{\lambda_{TN}}{x^1_{TN} - \bar{x}^1}\right)^{1/2}, \quad (108)$$

is actually independent of the coordinate $\bar{x}^1$ if the latter is enclosed in the above limits. Thus, near the toroidal surface, we have:

$$\Gamma(x^1) = \bar{\Gamma} + \varepsilon_{TN}\left(\frac{\lambda_{TN}}{x^1_{TN} - x^1}\right)^{1/2}.$$

The relationships obtained permit the formula of WKB approximation (106) near the toroidal surface to be represented as:

$$\Phi = -\sqrt{\pi\hat{k}^0_2\hat{\lambda}^0_{PN}}\tilde{\Phi}\left(\frac{\lambda_{TN}}{x^1_{TN} - x^1}\right)^{1/4}\left(\frac{p_0A}{pA_0}\right)^{1/2}$$

$$\times \exp\left[i\left(\bar{\psi} + \frac{\pi}{4}\right) - \bar{\Gamma} - 2i\left(\frac{x^1_{TN} - x^1}{\lambda_{TN}}\right)^{1/2} - \varepsilon_{TN}\left(\frac{\lambda_{TN}}{x^1_{TN} - x^1}\right)^{1/2}\right] r_N. \quad (109)$$

Using the asymptotic representation (99) we make sure that the solution (103) in the domain $(x^1_{TN} - x^1) \gg \lambda_{TN}$ functionally coincides with the solution (109), and by comparing them, it becomes possible to determine the constant D. After that, equation (103) assumes the form:

$$\Phi = -2i\hat{k}^0_2\hat{\lambda}^0_{PN}\tilde{\Phi}\,e^{i\bar{\psi} - \bar{\Gamma}}\left(\frac{p_0A}{pA_0}\right)^{1/2} g\left(\frac{x^1 - x^1_{TN}}{\lambda_{TN}} + i\varepsilon_{TN}\right) r_N. \quad (110)$$

Formulas (104), (106) and (110), taken together, describe a perturbed potential of the mode under investigation throughout the entire range of its existence, except for intervals lying deep in the opacity regions, i.e. when $x^1 - x^1_{TN} \gg \lambda_{TN}$ and when $x^1_{PN} - x^1 \gg \lambda_{PN}$. The solution in them is given by the WKB approximation [asymptotic values of k_{1N} are defined by the relationship (31)]. We do not write the corresponding formulas because the amplitude of the mode in these intervals is negligibly small, i.e. the oscillation in them can virtually be considered absent.

By knowing the perturbed electric potential Φ, it is easy to calculate the components of the perturbed electric field. Let us develop the expressions for the physical components $\hat{E}_i = E_i/\sqrt{g_i}$. We designate:

$$\tilde{E} = -i\bar{k}_2^0\tilde{\Phi},$$

being a typical value of the perturbed electric field. Near the poloidal surface, when $|x^1 - x^1_{PN}| \ll \Delta x^1_N$, we have:

$$\hat{E}_1 = -\frac{i\tilde{E}}{\bar{k}_2^0\bar{\lambda}_{PN}^0}\frac{p}{p_0}\left(\frac{\sigma_0 A}{\sigma A_0}\right)^{1/2} G'\left(\frac{x^1 - x^1_{PN}}{\lambda_{PN}} + i\varepsilon_{PN}\right) r_N,$$
$$\hat{E}_2 = \tilde{E}\left(\frac{\sigma_0 A}{\sigma A_0}\right)^{1/2} G\left(\frac{x^1 - x^1_{PN}}{\lambda_{PN}} + i\varepsilon_{PN}\right) r_N, \quad (111)$$

where $\sigma = \sqrt{g_\perp} = \sqrt{g_1 g_2}$.

Between the resonance surfaces, when $x^1 - x^1_{PN} \gg \lambda_{PN}$ and $x^1_{TN} - x^1 \gg \lambda_{TN}$, where the WKB approximation holds:

$$\hat{E}_1 = -\sqrt{\pi}\tilde{E}\left(\frac{v^1_{PN}}{v^1_N}\frac{pk^2_{1N}}{pk^2_{1N} + p^{-1}k_2^2}\frac{\sigma_0 A}{\sigma A_0}\right)^{1/2} \times \exp\left(i\int_{x^1_{PN}}^{x^1} k_{1N}\,dx^{1'} - \int_{x^1_{PN}}^{x^1}\frac{\gamma_N}{v^1_N}dx^{1'} + i\frac{\pi}{4}\right) r_N,$$
$$\hat{E}_2 = -\sqrt{\pi}\tilde{E}\left(\frac{v^1_{PN}}{v^1_N}\frac{p^{-1}k_2^2}{pk^2_{1N} + p^{-1}k_2^2}\frac{\sigma_0 A}{\sigma A_0}\right)^{1/2} \times \exp\left(i\int_{x^1_{PN}}^{x^1} k_{1N}\,dx^{1'} - \int_{x^1_{PN}}^{x^1}\frac{\gamma_N}{v^1_N}dx^{1'} + i\frac{\pi}{4}\right) r_N. \quad (112)$$

Near the toroidal surface, when $|x^1 - x^1_{TN}| \ll \Delta x^1_N$:

$$\hat{E}_1 = -2\frac{\lambda_{PN}}{\lambda_{TN}}\tilde{E}\left(\frac{\sigma_0 A}{\sigma A_0}\right)^{1/2} e^{i\bar{\psi} - \bar{\Gamma}} g'\left(\frac{x^1 - x^1_{TN}}{\lambda_{TN}} + i\varepsilon_{TN}\right) r_N,$$
$$\hat{E}_2 = -2i\bar{k}_2^0\bar{\lambda}_{PN}^0\frac{p_0}{p}\tilde{E}\left(\frac{\sigma_0 A}{\sigma A_0}\right)^{1/2} e^{i\bar{\psi} - \bar{\Gamma}} g\left(\frac{x^1 - x^1_{TN}}{\lambda_{TN}} + i\varepsilon_{TN}\right) r_N. \quad (113)$$

From these expressions it is directly evident how, in going from the poloidal to the toroidal surface, the polarization of the mode changes from poloidal to toroidal. The amplitude of the perturbed electric field varies from a value of order $\tilde{E}$ on the poloidal surface to a value of order $(\lambda_{PN}/\lambda_{TN})\,e^{-\bar{\Gamma}}\,\tilde{E}$ on the toroidal surface. The factor $(\lambda_{PN}/\lambda_{TN}) = (v^1_{PN}/v^1_{TN})^{1/2}$, equal, on the order of magnitude, to $(\alpha_N m)^{4/3} \gg 1$, describes the increase in wave amplitude due to a decrease of the typical value of group velocity in the toroidal (compared with that in the poloidal) surface. The factor $e^{-\bar{\Gamma}}$ describes the damping of the wave as a consequence of the dissipation on the ionosphere. On the order of magnitude, $\bar{\Gamma} \sim m(\gamma_N/\omega)$. If there is an instability that is stronger than the dissipation in the ionosphere, then $\bar{\Gamma} < 0$. In this case, the wave amplitude in the neighbourhood of the toroidal surface is able to increase quite substantially.

Near the poloidal surface, even if the damping is totally neglected (i.e. when $\varepsilon_{PN} = 0$), the wave field remains a finite one in magnitude, namely the function $G[(x^1 - x^1_{PN})/\lambda_{PN}]$ and its derivatives are regular. This means that the transverse dispersion of the wave has time to pump out the energy supplied by the source (extraneous currents in the ionosphere). The transverse group velocity, though going to zero on the poloidal surface, does so rather slowly, as $(x^1 - x^1_{PN})^{1/2}$. On the contrary, on the toroidal surface, when $\varepsilon_{TN} = 0$, the electric field has a singularity, $\hat{E}_1 \sim (x^1 - x^1_{TN})^{-1}$. On this surface the group velocity goes to zero quickly, as $(x^1_{TN} - x^1)^{3/2}$. With the small damping taken into account, the field singularity is regularized:

$$\hat{E}_1 \sim \frac{1}{x^1 - x^1_{TN} + i\varepsilon_{TN}\lambda_{TN}}.$$

By knowing the perturbed potential Φ, using formulas (6), (7) and (8) one can find the wave's perturbed magnetic field. Its longitudinal structure is described by the function $\partial R_N/\partial \ell$. From the same considerations as when introducing the function r_N, we introduce in our treatment the function:

$$b_N = \frac{(q_N A t_A)^{1/2}}{\omega}\frac{\partial R_N}{\partial \ell}.$$

It is easy to see that it satisfies the relationship

$$\langle b_N^2\rangle = 1.$$

Near the poloidal and the toroidal surface, respectively,

$$b_N = \frac{1}{\omega}\left(\frac{At_A}{p}\right)^{1/2}\frac{\partial P_N}{\partial \ell}, \qquad b_N = \frac{(pAt_A)^{1/2}}{\omega}\frac{\partial T_N}{\partial \ell}.$$

For large N:

$$b_N = \sqrt{2}\cos\left(\frac{2\pi N}{t_A}\int_l^{\ell}\frac{d\ell'}{A}\right).$$

We introduce also the typical value of perturbed magnetic field:

$$\tilde{B} = i(c/A_0)\tilde{E} = (\bar{k}_2^0 c/A_0)\tilde{\Phi}.$$

Using these notations we write down the physical components $\hat{B}_i = B_i/\sqrt{g_i}$. When $|x^1 - x^1_{PN}| \ll \Delta x^1_N$:

$$\hat{B}_1 = \tilde{B}\left(\frac{\sigma_0 A}{\sigma A_0}\right)^{1/2} G\left(\frac{x^1 - x^1_{PN}}{\lambda_{PN}} + i\varepsilon_{PN}\right) b_N,$$
$$\hat{B}_2 = \frac{i\tilde{B}}{\bar{k}_2^0\bar{\lambda}_{PN}^0}\frac{p}{p_0}\left(\frac{\sigma_0 A}{\sigma A_0}\right)^{1/2} G'\left(\frac{x^1 - x^1_{PN}}{\lambda_{PN}} + i\varepsilon_{PN}\right) b_N. \quad (114)$$

When $x^1 - x^1_{PN} \gg \lambda_{PN}$ and $x^1_{TN} - x^1 \gg \lambda_{TN}$:

$$\hat{B}_1 = -\sqrt{\pi}\tilde{B}\left(\frac{v^1_{PN}}{v^1_N}\frac{p^{-1}k_2^2}{pk^2_{1N} + p^{-1}k_2^2}\frac{\sigma_0 A}{\sigma A_0}\right)^{1/2} \quad (115a)$$
$$\times \exp\left(i\int_{x^1_{PN}}^{x^1} k_{1N}\,dx^{1'} - \int_{x^1_{PN}}^{x^1}\frac{\gamma_N}{v^1_N}dx^{1'} + i\frac{\pi}{4}\right) b_N,$$

$$\hat{B}_2 = \sqrt{\pi}\tilde{B}\left(\frac{v^1_{PN}}{v^1_N}\frac{pk^2_{1N}}{pk^2_{1N}+p^{-1}k^2_2}\frac{\sigma_0 A}{\sigma A_0}\right)^{1/2}$$
$$\times \exp\left(\mathrm{i}\int_{x^1_{PN}}^{x^1} k_{1N}\,\mathrm{d}x^{1'} - \int_{x^1_{PN}}^{x^1}\frac{\gamma_N}{v^1_N}\mathrm{d}x^{1'} + \mathrm{i}\frac{\pi}{4}\right)b_N. \quad (115\mathrm{b})$$

When $|x^1 - x^1_{TN}| \ll \Delta x^1_N$:

$$\hat{B}_1 = -2\mathrm{i}\hat{k}^0_2\hat{\lambda}^0_{PN}\frac{p_0}{p}\tilde{B}\left(\frac{\sigma_0 A}{\sigma A_0}\right)^{1/2} \mathrm{e}^{\mathrm{i}\bar{\psi}-\bar{\Gamma}} g\left(\frac{x^1 - x^1_{TN}}{\lambda_{TN}} + \mathrm{i}\varepsilon_{TN}\right)b_N,$$
$$\hat{B}_2 = 2\frac{\lambda_{PN}}{\lambda_{TN}}\tilde{B}\left(\frac{\sigma_0 A}{\sigma A_0}\right)^{1/2} \mathrm{e}^{\mathrm{i}\bar{\psi}-\bar{\Gamma}} g'\left(\frac{x^1 - x^1_{TN}}{\lambda_{TN}} + \mathrm{i}\varepsilon_{TN}\right)b_N. \quad (116)$$

12. Some remarks on the comparison of theory with experiment

Detailed comparisons of the theory presented here with observational data on hydromagnetic magnetospheric oscillations (ULF waves) are beyond the scope of this paper and must be the subject of special investigation. In the context of such an investigation it is necessary, in particular, to explore the possibility of deriving fine transverse structure of the Alfvén oscillations on the basis of existing observational data as well as to propose new experiments aimed at studying the fine structure. Such an analysis would also require a further development of the theory, namely the transition from monochromatic to broad-banded oscillations and from separate harmonics in azimuth to their superposition. Yet, we consider it appropriate to make some simple remarks.

One of the main theoretical conclusions is the establishment of the fact that a purely poloidal mode of Alfvénic oscillations in the magnetosphere is nonexistent.

The oscillations with large values of m, considered in this paper, are excited as poloidal ones (the disturbed magnetic field oscillates in the radial direction), but as they travel across the magnetic shells their radial spatial structure rapidly becomes fine, which, in accordance with the polarization properties of the Alfvén wave, makes them lose their poloidal character. As a result, throughout most of the transparency region the oscillation has radial and azimuthal components of the magnetic field, comparable in magnitude, while near the toroidal surface, predominantly the azimuthal component. This fact is also supported by satellite observations of the poloidal oscillations of the magnetosphere; usually, in addition to the predominant radial component of the oscillations of a disturbed magnetic field, an azimuthal component of a comparable magnitude is recorded (Anderson *et al.*, 1990; Takahashi *et al.*, 1990; Takahashi and Anderson, 1992).

Oscillations with small values of m that are generated, for example, through field-line resonance, have from the outset a toroidal character, and fine structuring of their radial spatial structure in the process of their propagation only enhances their toroidal character (Southwood, 1974; Chen and Hasegawa, 1974).

These theoretical conclusions are associated naturally with the known experimental fact: about half of the observed Alfvén oscillations are composed of nonstructured oscillations, about 30% correspond to toroidal oscillations, 10% are made up by oscillations with a large proportion of the compressible component, and only 5% refer to poloidal oscillations with the predominant radial component (Anderson *et al.*, 1990). This is quite understandable in the context of the above theory. A purely poloidal oscillation can be recorded only in a narrow neighbourhood near the poloidal surface where, however, only the monochromatic oscillation will be a poloidal one. If the disturbance is a broad-banded one, then the surface, a resonance one for a given frequency, will not be a resonance one for another frequency, which further reduces the probability of detecting poloidal oscillations.

An other remark concerns hodographs for magnetic field disturbances and plasma velocity.

The behaviour of the hodograph in different areas inside of the transparency region, constructed in accordance with formulas (114)–(116), is shown in Fig. 7. An interesting behavioural feature of the hodograph is the reversal of the rotation direction of the disturbance vector as one moves from the poloidal to toroidal resonance surface. For positive k_2 near the poloidal surface, the vector of the disturbed magnetic field and the plasma velocity rotates clockwise, and near the toroidal surface it rotates anti-

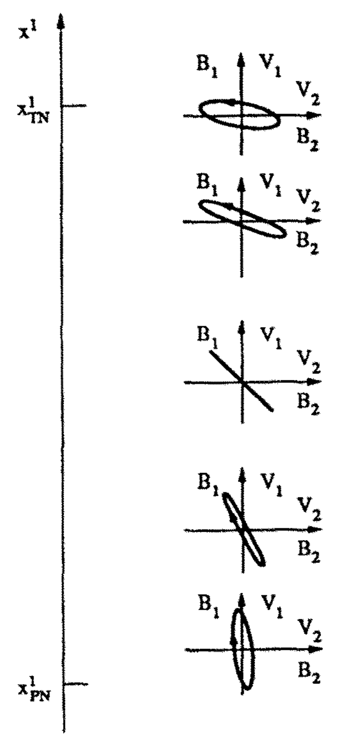

Fig. 7. Hodographs of monochromatic transverse small-scale Alfvén oscillations at different points inside the transparency region. To ease the comparison with observations, the hodographs are constructed in the planes (B_1, B_2) and (v_1, v_2). The most interesting peculiarities are a change in orientation of the polarization ellipse when moving from the poloidal point of turn to the toroidal point, and the rotation reversal of the hodograph

clockwise. In between the resonance surfaces, that is, in most of the transparency region, the oscillations are linearly polarized.

Such behaviour of the hodograph can serve as the indicator of the oscillations under consideration.

These results can be compared with the coordinated observations on satellite *GEOS* and in the conjugated (along the geomagnetic field line) ionospheric *E*-region by the STARE radar (Walker *et al.*, 1982). It should be noted that such a comparison should be carried out with some caution. The point here is that the oscillations recorded on the satellite included the predominant part of the compressible component, in addition to which a poloidal-type oscillation was recorded. Besides, the period of the oscillations varied in the course of the observation from 213 to 300 s, which indicates a broad-banded character of the oscillations. It might, however, be anticipated that instantaneous values of the recorded parameters correspond to the parameters of the monochromatic wave recorded at a given point. As far as the larger contribution of the compressible component is concerned, on the terrestrial surface it does not seem to be recorded; therefore, the data from the STARE radar can be compared with the results of our theory. Walker *et al.* (1982) give polarization ellipses for oscillations constructed based on observations of the plasma motions in a 40×40 km square in the ionospheric *E*-region. If it is assumed that the observational square is located as shown in Fig. 2 of this paper, then the behaviour of the hodographs in Fig. 5 from the cited paper, in view of the data in Fig. 4 from the same paper, agrees quite well with our theoretical picture.

The last remark is on the localization of the monochromatic poloidal oscillations. In observations on satellites, nearly monochromatic oscillations of the magnetic field components are recorded in rather narrow ranges of the magnetic shells (Engebretson *et al.*, 1988; Anderson *et al.*, 1990; Takahashi *et al.*, 1990). The width of these ranges in the equatorial region, 1–$3R_E$, agrees well with the distance between the poloidal and toroidal resonance surfaces for the second harmonic obtained in our numerical model. We wish to stress that it is just the second harmonic which has an antinode at the equator, unlike the first and third harmonics which have a node in magnetic field there.

13. Conclusion

Let us formulate the main results of this work.

(1) Based on the equation of ideal magnetic hydrodynamics we have obtained a partial differential equation (4) which, together with the boundary condition on the ionosphere (12), describes the scalar potential of monochromatic transversally small-scale ($m \gg 1$) standing Alfvén waves in the axisymmetric magnetosphere.

(2) It has been shown that the WKB approximation in coordinate x^1 normal to magnetic surfaces, is applicable for solving equation (4). Using this approximation we have formulated a one-dimensional (in longitudinal coordinate ℓ) boundary-value problem for eigenvalues [equations (15), (16)]. The eigenfunctions of this problem $R_N(x^1, \ell, \omega)$ that represent standing waves with N nodes on the field line, describe the longitudinal structure of the oscillation field, and eigenvalues of $k_{1N}(x^1, \omega)$ are eigenvalues of the quasiclassical wavevector. They depend on the coordinate x^1 and frequency ω as the parameters. For harmonics with large longitudinal wavenumbers, $N \gg 1$, we have obtained equation (42) relating k_{1N} to x^1 and ω and to the explicit expression (44) for R_N.

(3) We have introduced the notions of the poloidal and the toroidal resonance surfaces, on which the quasiclassical wavevector $k_{1N}(x^1, \omega)$ goes, respectively, to zero and to infinity, and the polarization of the oscillations has a strictly poloidal and toroidal character. Definitions were given to the poloidal, $\Omega_N^P(x^1)$, and the toroidal, $\Omega_N^T(x^1)$, eigenfrequencies as the frequencies of such oscillations which on a given magnetic surface x^1 are, respectively, poloidal and toroidal ones. It has been found that, except for small neighbourhoods of the extrema of the functions $\Omega_N^P(x^1)$ and $\Omega_N^T(x^1)$, the transparency region, $k_{1N}^2(x^1, \omega) > 0$, of the mode with a given frequency lies between the poloidal and toroidal resonance surfaces. It has been shown that, as a consequence of the curvature of geomagnetic field lines, the Alfvén wave under study propagates (even in the approximation of ideal MHD) across the magnetic shells. The expressions (33) have been obtained for transverse components of group velocity.

(4) In the next order of the WKB approximation we have obtained the expression (68) which describes the variation in oscillation amplitude across the magnetic shells. This variation is due both to a change in transverse group velocity and to the dissipation of the wave in the ionosphere. If an instability of the oscillations involved occurs in the magnetosphere, then their amplitude can increase in the course of the propagation.

(5) The validity range of the WKB approximation is violated near the toroidal and poloidal magnetic surfaces which are a usual turning point and a singular turning point in coordinate x^1. The solution in the neighbourhoods of these surfaces has been found in terms of perturbation theory based on the closeness of the oscillations under investigation to the poloidal and toroidal modes near the corresponding surfaces. It has been shown that the mode generated near the poloidal surface by extraneous currents in the ionosphere, after propagating through the transparency region, is totally absorbed in the neighbourhood of the toroidal surface. A matching of solutions in different regions has been carried out, as well as obtaining formulas (111)–(116) describing the global structure of the oscillations concerned.

(6) For a magnetospheric model involving a dipole model of the geomagnetic field and the model (51) of the Alfvén velocity distribution, the longitudinal problem for eigenvalues [equations (15), (16)] has been solved. The qualitative results obtained in this paper have been shown to be in full agreement with results of numerical calculations.

(7) It is shown that some of the simplest results of the theory presented here agree quite well with experimental evidence.

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