A MODEL OF SOLAR DIFFERENTIAL ROTATION

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Abstract. An axisymmetric model for the Sun's differential rotation based upon a mechanism for angular momentum transport by compressible convection is developed. Convective heat transport is also considered. The model is simplified by the neglect of meridional circulation and radiative heat transport but is otherwise a self-consistent one because no adjustable parameters are used and the effective transport coefficients are expressed explicitly in terms of superadiabaticity of stratification rather than assigned externally. The model predictions agree satisfactorily with the observed rotation of the photosphere and with helioseismology data. The dependence of the rotation law, produced by the model, on rotation rate of a Sun-like star is discussed.

1. Introduction

This paper generalizes the mechanism for angular momentum transport suggested earlier (Kichatinov, 1987; henceforth Paper I). Next, the (numerical) model for solar differential rotation is developed based on this mechanism. The model treats convective transport of both angular momentum and heat.

Mixing-length arguments and various simplifying assumptions based on qualitative considerations only are used, the price paid for self-consistency. The model uses essentially no adjustable parameters. For example, effective heat conductivity (tensoral) is expressed in terms of superadiabatic temperature gradient rather than assigned externally. Anisotropy of turbulent mixing is treated as an explicit function of the local angular velocity. In other words, the model suggested may be named semi-phenomenological but self-consistent.

The basic equations of the model are nonlinear ones and are solved numerically. The solution yields angular velocity and potential temperature distributions with radius and latitude. The input parameters for the model are convection zone depth, fluid density at the convection zone base, radius, luminosity, surface gravity (mass), and mean angular velocity of a star.

Another essential point is allowance for fluctuating magnetic fields of turbulent origin. Magnetic Reynolds numbers for solar convection are known to be large. Under this condition, turbulent (convective) fluids must be unstable to fluctuating seed magnetic fields, i.e., magnetic inhomogeneities of the scale of convective cells and smaller (Kraichnan and Nagarajan, 1967; Kazantsev, 1968; Pouquet, Frisch, and Leorat, 1976; Meneguzzi, Frisch, and Pouquet, 1981.) The growth of magnetic inhomogeneities is stabilized, probably, near equipartition of kinetic and magnetic energies, at least in the inertial range of wave numbers (Pouquet, Frisch, and Leorat, 1976; Meneguzzi, Frisch, and Pouquet, 1981). Observations of fine-structured magnetic fields of kilogauss strength suggest that the small-scale dynamo does, indeed, operate in the Sun. In this

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case, Maxwell stresses must be taken into account in addition to the Reynolds stresses usually considered by differential rotation theories.

2. Maxwell Stresses and Turbulent Angular Momentum Transport

The off-diagonal components, $T_{r\theta}$ and $T_{\theta\varphi}$ (r, θ, and φ are usual spherical coordinates), of a stress tensor are proportional to zonal momentum fluxes and are of major importance for the differential rotation problem (Rüdiger, 1989). Reynolds stresses, $R_{ij} = -\rho \langle u_i u_j \rangle$, are usually considered; angular brackets signify averaging, and $\mathbf{u}$ is fluctuating velocity. If, however, near-equipartition of energy of fluctuating magnetic, $\mathbf{h}$, and velocity fields takes place, the Maxwell stresses, $M_{ij} = \langle h_i h_j - \delta_{ij} h/2 \rangle / \mu$, are of the same order of magnitude as the Reynolds stresses.

It should be noted at this point that the field $\mathbf{h}$ is generated efficiently by a small-scale dynamo (Kraichnan and Nagarajan, 1967; Kazantsev, 1968; Pouquet, Frisch, and Leorat, 1976; Meneguzzi, Frisch, and Pouquet, 1981), and the mean magnetic field is not needed for this process. We shall neglect the latter field and shall assume statistical properties of the fluctuations, $\mathbf{h}$, to be steady.

Defining these properties is required to derive the tensor M_{ij}. Unfortunately, this could not be done in a strict way. A rigorous description of solar magnetoconvection remains still beyond the reach of present-day theories. We shall be using some simplifying assumptions which can be justified only qualitatively.

Using the mixing-length approximation results in the following relation for the spectral tensor of momentum density $\boldsymbol{\rho} = \rho\mathbf{u}$ for a rotating fluid (Kichatinov, 1986):

$$\langle \hat{\rho}_i(\mathbf{z}) \hat{\rho}_j(\mathbf{z}') \rangle = B_{im}(\omega, \mathbf{z}) B_{jn}(\omega, \mathbf{z}') \langle \hat{\rho}^o_m(\mathbf{z}) \hat{\rho}^o_n(\mathbf{z}') \rangle , \qquad (2.1)$$

where circumflexes mean Fourier-amplitudes, e.g.,

$$\hat{\boldsymbol{\rho}}(\mathbf{z}) = \int \exp(-i\mathbf{z}\cdot\mathbf{r}) \boldsymbol{\rho}(\mathbf{r}) \, d\mathbf{r}/(2\pi)^3 ;$$

$\omega = 2\tau\Omega$ is the Coriolis number (reciprocal of the Rossby number) with τ and Ω being the convective turnover time and angular velocity respectively; the upper index 'o' in (2.1) corresponds to the so-called original turbulence (Rüdiger, 1989) which would take place under actual sources of turbulence but in the absence of rotation; tensor B_{im} is defined as

$$B_{im}(\omega, \mathbf{z}) = (\delta_{im} + \omega\sigma\varepsilon_{imp} z_p / z)/(1 + \omega^2\sigma^2) , \qquad (2.2)$$

where $\sigma = (\mathbf{z}\cdot\boldsymbol{\Omega})/(z\Omega)$ is a cosine of the angle between the wave vector $\mathbf{z}$ and angular velocity $\boldsymbol{\Omega}$. Equation (2.1) defines the influence of rotation on turbulence and allows the spectral tensor for rotating fluids to be found if the properties of the original turbulence are known. The momentum density, $\boldsymbol{\rho} = \rho\mathbf{u}$, is convenient to use in (2.1) and below because the field $\boldsymbol{\rho}$ but not the velocity $\mathbf{u}$ is divergence-free for subsonic convection of a density-stratified fluid (Ogura and Phillips, 1962; Gilman and Glatzmaier, 1981).

The spectral tensor of magnetic fields $\mathbf{h}$ will be assumed to have the same structure as the spectral tensor for the motions generating these fields, i.e., to satisfy Equations (2.1):

$$\langle \hat{h}_i(\mathbf{z})\hat{h}_j(\mathbf{z}')\rangle = B_{im}(\omega, \mathbf{z})B_{jn}(\omega, \mathbf{z}')\,\langle \hat{h}^0_m(\mathbf{z})\hat{h}^0_n(\mathbf{z}')\rangle\,. \tag{2.3}$$

As in Paper I, the original turbulence will be assumed quasi-isotropic:

$$\langle \hat{h}^0_m(\mathbf{z})h^0_n(\mathbf{z}')\rangle = \frac{\hat{H}(\boldsymbol{\kappa}, k)}{8\pi k^2}\,[\delta_{mn} - (1 + \kappa^2/4k^2)k_m k_n/k^2 + \\ + (\kappa_m k_n - \kappa_n k_m)/2k^2 + \kappa_m\kappa_n/4k^2]\,, \tag{2.4}$$

$$\boldsymbol{\kappa} = \mathbf{z} + \mathbf{z}'\,, \qquad \mathbf{k} = (\mathbf{z} - \mathbf{z}')/2\,,$$

where $\hat{H}$ is the Fourier-transform of the local spectrum, i.e., the function

$$H(\mathbf{r}, k) = \int \exp(i\boldsymbol{\kappa}\cdot\mathbf{r})\hat{H}(\boldsymbol{\kappa}, k)\,\mathrm{d}\boldsymbol{\kappa}$$

is the local spectrum:

$$\langle h^2\rangle^0 = \int_0^\infty H(\mathbf{r}, k)\,\mathrm{d}k\,.$$

Assume next the energy equipartition,

$$\langle h^2\rangle^0/\mu = \rho\,\langle u^2\rangle^0\,. \tag{2.5}$$

The above assumptions suffice to derive the Maxwell stresses. Let us consider first the nondiffusive part, $\Lambda^{(m)}_{ij}$, of the correlation tensor $Q^{(m)}_{ij} = -\langle h_i h_j\rangle/\mu\rho$ which is found when spatial inhomogeneity of angular velocity is neglected. The appearance of the nondiffusive stresses for rotating fluids has been named the Λ-effect (Rüdiger, 1989). The Λ-effect prevents the rigid-body rotation from being a solution of the equation of motion. In other words, the nondiffusive correlations, Λ_{ij}, represent sources of differential rotation. Expressions for the desired off-diagonal components of the $\Lambda^{(m)}$-tensor follow from substitution of (2.4) into (2.3) with subsequent transformation to real correlations and using (2.5). We find

$$\Lambda^{(m)}_{r\varphi} = -\omega l^2\,\langle u^2\rangle^0 \sin\theta(\Psi_1(\omega) - \Psi_2(\omega)\cos^2\theta)\,\frac{r}{\rho}\,\frac{\mathrm{d}}{\mathrm{d}r}\,\frac{1}{r}\,\frac{\mathrm{d}\rho}{\mathrm{d}r}\,, \\ \Lambda^{(m)}_{r\theta} = -\omega l^2\,\langle u^2\rangle^0\,\Psi_2(\omega)\cos\theta\sin^2\theta\,\frac{r}{\rho}\,\frac{\mathrm{d}}{\mathrm{d}r}\,\frac{1}{r}\,\frac{\mathrm{d}\rho}{\mathrm{d}r}\,, \tag{2.6}$$

where

$$l^2 = \int_0^\infty H(r, k)k^{-2}\,\mathrm{d}k/\langle h^2\rangle^0$$

is a typical scale of fluctuations, and the functions $\Psi_1(\omega)$ and $\Psi_2(\omega)$ are

$$\Psi_1(\omega) = \frac{1}{8\omega^4}\left[2 + \frac{1}{1+\omega^2} - \frac{3}{\omega}\tan^{-1}(\omega)\right],$$

$$\Psi_2(\omega) = \frac{1}{16\omega^4}\left[30 - \omega^2 + \frac{2\omega^4}{1+\omega^2} + \frac{\omega^4 - 9\omega^2 - 30}{\omega}\tan^{-1}(\omega)\right]. \tag{2.7}$$

It has been taken into account when deriving (2.6) that the quantity $l^2 \langle u^2 \rangle^0$ varies with r much less than the density ρ does (Spruit, 1974; Gough and Weiss, 1976).

By adding the nondiffusive part of the velocity correlation tensor $\langle u_i u_j \rangle$ found in Paper I to (2.6), we find the full (usual plus magnetic) Λ-effect:

$$\Lambda_{r\varphi} = \omega \langle u^2 \rangle^0 \sin\theta (\Psi_1(\omega) - \Psi_2(\omega)\cos^2\theta)\, G\,,$$

$$\Lambda_{r\theta} = \omega \langle u^2 \rangle^0\, \Psi_2(\omega) \cos\theta \sin^2\theta\, G\,, \tag{2.8}$$

where the dimensionless quantity

$$G = l^2 \left(\frac{r}{\rho^2} \frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{r} \frac{\mathrm{d}\rho^2}{\mathrm{d}r} - \frac{r}{\rho} \frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{r} \frac{\mathrm{d}\rho}{\mathrm{d}r} \right) \tag{2.9}$$

is related to density inhomogeneity. (Note that the correlations Λ_{ij} contribute to the stress tensor T_{ij} with a minus sign, $T_{ij} = -\rho\Lambda_{ij} + \cdots$.)

The second term in (2.9) is involved through allowance for fluctuating magnetic fields. This contribution decreases the value of G for the near-adiabatically stratified convection zone, i.e., magnetic fluctuations weaken the Λ-effect. Another way in which the small-scale magnetic fields decrease differential rotation is an increase of effective viscosities, as discussed below. Nevertheless, we shall see in Section 5 that the remaining Λ-effect suffices to produce differential rotation of the solar value.

Let a pseudo-vector $\boldsymbol{\lambda}$ be the nondiffusive flux of zonal momentum: $\lambda_r = \rho\Lambda_{r\varphi}$ and $\lambda_\theta = \rho\Lambda_{\theta\varphi}$. If rotation is slow ($\omega \ll 1$), we find from (2.7) $\Psi_1 = \frac{1}{20}$, $\Psi_2 = 0$, and

$$\boldsymbol{\lambda} = \mathbf{e}_r \omega \langle u^2 \rangle^0 \sin\theta\, G/20\,, \tag{2.10}$$

where $\mathbf{e}_r$ is a radial unit vector. Therefore, the flux $\boldsymbol{\lambda}$ is purely radial in the slow rotation case. In the opposite limit of rapid rotation ($\omega \gg 1$) Equations (2.7) yield $\Psi_1 = 0$, $\Psi_2 = \pi/32\,|\omega|$ (only first-order terms in ω^{-1} are kept). Hence, for the rapid rotation case we find

$$\boldsymbol{\lambda} = -\mathbf{e}\pi \langle u^2 \rangle^0 \sin\theta\cos\theta\, G/32\,, \tag{2.11}$$

where $\mathbf{e} = \mathbf{\Omega}/\Omega$ is the unit vector parallel to the rotation axis. The flux (2.11) transports zonal momentum towards the equatorial plane and is parallel to the axis of rotation. For intermediate values of Coriolis number, ω, the flux $\boldsymbol{\lambda}$ rotates from the radial direction towards the equatorial plane as ω increases. Note that there is no dependence of (2.11) on angular velocity. This is because the (density) inhomogeneity of convection zone is considered in this paper to be a cause for the Λ-effect. If convection anisotropy were considered as the cause, the flux $\boldsymbol{\lambda}$ would be proportional to ω^{-2} in the case of $\omega \gg 1$ (Rüdiger, 1983).

Besides the Λ-effect, rotating turbulence produces the ordinary effective diffusion. The diffusion results from the reaction of turbulence on the inhomogeneity of a mean flow. Mixing-length derivations of diffusive contributions to the Reynolds stresses with account for global rotation were made by Kichatinov (1986). Let us derive the diffusive part, $D_{ij}^{(m)}$, of the Maxwell stresses.

The derivation needs the perturbations $\mathbf{h}'$ of fluctuating magnetic fields by deviation of rotation from homogeneity to be accounted for. Let the deviation be characterized by the velocity $\mathbf{V}$. We adopt a simple approximation equivalent to the mixing-length approach. The part $\partial \mathbf{h}/\partial t - \nabla \times (\mathbf{u} \times \mathbf{h})$ of the induction equation will be replaced by the τ-relaxation term, $\mathbf{h}'/\tau$, where $\tau \approx l/u$ is the lifetime of a convective eddy. This implies that nonlinear interactions in the turbulence cause the perturbations $\mathbf{h}'$ to relax in a typical time τ. Then, we find

$$\mathbf{h}' = \tau \nabla \times (\mathbf{V} \times \mathbf{h}) . \tag{2.12}$$

The diffusive part of the Maxwell stress tensor is

$$D_{ij}^{(m)} = (\langle h_i h_j' \rangle + \langle h_i' h_j \rangle - \delta_{ij} \langle \mathbf{h} \cdot \mathbf{h}' \rangle)/\mu .$$

By using (2.3)–(2.5) and (2.12), we find to the first order in the ratio l/l', where l' is the scale of mean flow,

$$\begin{aligned} D_{r\varphi}^{(m)} &= \rho \nu_{\perp}^{(m)} r \sin\theta \, \partial\Omega/\partial r + \rho(\nu_{\parallel}^{(m)} - \nu_{\perp}^{(m)}) \times \\ &\quad \times \sin\theta \cos\theta (r \cos\theta \, \partial\Omega/\partial r - \sin\theta \, \partial\Omega/\partial\theta) , \\ D_{\theta\varphi}^{(m)} &= \rho \nu_{\perp}^{(m)} \sin\theta \, \partial\Omega/\partial\theta + \rho(\nu_{\parallel}^{(m)} - \nu_{\perp}^{(m)}) \times \\ &\quad \times \sin^2\theta (\sin\theta \, \partial\Omega/\partial\theta - r \cos\theta \, \partial\Omega/\partial r) . \end{aligned} \tag{2.13}$$

$\nu_{\parallel}^{(m)}$ and $\nu_{\perp}^{(m)}$ are effective viscosities caused by magnetic inhomogeneities for the directions parallel and normal to the rotation axis, respectively,

$$\begin{aligned} \nu_{\parallel}^{(m)} &= \tau \frac{\langle h^2 \rangle^0}{\mu\rho} (a + b) = \tau \langle u^2 \rangle^0 (a + b) , \\ \nu_{\perp}^{(m)} &= \tau \frac{\langle h^2 \rangle^0}{\mu\rho} a = \tau \langle u^2 \rangle^0 a , \end{aligned} \tag{2.14}$$

where the equipartition relation (2.5) was used, and the quantities a and b are functions of the Coriolis number:

$$\begin{aligned} a(\omega) &= \frac{1}{4\omega^2} \left[1 + \frac{\omega^2 - 1}{\omega} \tan^{-1}(\omega) \right] , \\ \mathrm{b}(\omega) &= \frac{1}{4\omega^2} \left[\frac{3 + \omega^2}{\omega} \tan^{-1}(\omega) - 3 \right] . \end{aligned} \tag{2.15}$$

The effective viscosities (2.14) are brought about by resistance of magnetic inhomogeneities to deformation by inhomogeneous rotation. Magnetic contributions (2.14) increase effective viscosities and cancel the negativity of $\nu_\perp$ at large Coriolis numbers found when magnetic fields were neglected (Kichatinov, 1986).

The Maxwell stress tensor is the sum of diffusive (2.13) and nondiffusive (2.6) contributions:

$$M_{ij} = D_{ij}^{(m)} - \rho A_{ij}^{(m)} . \tag{2.16}$$

The next step is to find an expression for $\langle u^2 \rangle^0$. As has been mentioned before, $\langle u^2 \rangle^0$ is the intensity of the so-called original turbulence which would take place under actual sources of turbulence (actual superadiabaticity), but if rotation were absent. The Sun, however, rotates and original turbulence does not exist in reality. Therefore, the quantity $\langle u^2 \rangle^0$ is not measurable (observable) and it is desirable to express $\langle u^2 \rangle^0$ in (2.8) and (2.14) in terms of some other parameters. In accordance with the definition of original turbulence, the relations valid for nonrotating fluids may be used for this aim. The known mixing-length relation will be adopted:

$$\langle u^2 \rangle^0 = - \frac{l^2 g}{4T} \frac{\partial \Delta T}{\partial r} , \tag{2.17}$$

where T is mean temperature, g is gravity, and ΔT is the potential temperature (deviation of T from the adiabatic profile T_a, $\Delta T = T - T_a$), i.e., $\partial \Delta T/\partial r = \partial T/\partial r + g/c_p$ is the superadiabatic gradient. The equation for ΔT will be derived below and used in the model developed to define $\langle u^2 \rangle^0$ using (2.17). The turbulent stress tensor T_{ij} is composed from Reynolds and Maxwell stresses: $T_{ij} = R_{ij} + M_{ij}$. By using (2.8), (2.13), relation (2.16), and the Reynolds stresses written in Paper I, we find the following expressions for the off-diagonal components $T_{j\varphi}$ $(j = r, \theta)$ of the turbulent stress tensor:

$$T_{r\varphi} = - \rho\tau \frac{l^2 g}{4T} \frac{\partial \Delta T}{\partial r} \sin\theta \left[\Psi_3(\omega) r \frac{\partial \Omega}{\partial r} + \Psi_4(\omega) \cos\theta \times \right.$$
$$\left. \times \left(r \cos\theta \frac{\partial \Omega}{\partial r} - \sin\theta \frac{\partial \Omega}{\partial \theta} \right) - 2\Omega G (\Psi_1(\omega) - \Psi_2(\omega) \cos^2\theta) \right] ,$$
$$(2.18)$$
$$T_{\theta\varphi} = - \rho\tau \frac{l^2 g}{4T} \frac{\partial \Delta T}{\partial r} \sin\theta \left[\Psi_3(\omega) \frac{\partial \Omega}{\partial \theta} + \Psi_4(\omega) \sin\theta \times \right.$$
$$\left. \times \left(\sin\theta \frac{\partial \Omega}{\partial \theta} - r \cos\theta \frac{\partial \Omega}{\partial r} \right) - 2\Omega G \Psi_2(\omega) \sin\theta \cos\theta \right] ,$$

where Ψ_1, Ψ_2, and G where defined by (2.7) and (2.9), and the functions $\Psi_3(\omega)$ and

$\Psi_4(\omega)$ are

$$\Psi_3(\omega) = \frac{1}{32\omega^4}\left[\frac{8\omega^4}{1+\omega^2} - 3 - \omega^2 + \frac{3 + 2\omega^2 + 7\omega^4}{\omega}\tan^{-1}(\omega)\right],$$
$$\Psi_4(\omega) = \frac{1}{32\omega^4}\left[15 - 11\omega^2 - \frac{8\omega^4}{1+\omega^2} + \frac{13\omega^4 + 6\omega^2 - 15}{\omega}\tan^{-1}(\omega)\right]. \tag{2.19}$$

Note that the functions $\Psi_n(\omega)$ introduce an explicit nonlinear dependence on angular velocity into the stresses (2.18).

3. Convective Heat Fluxes

There is a very large literature on convective heat transport in rotating fluids (see, e.g., Rüdiger, 1989; and references therein). This paper adopts a simple mixing-length representation for convective heat flux $\mathbf{F} = c_p \rho \langle T' \mathbf{u} \rangle$ (T' is fluctuating temperature):

$$\mathbf{F} = -\rho c_p \tau \langle \mathbf{u} u_j \rangle \nabla_j \Delta T, \tag{3.1}$$

which was brought in use by Wasiutynski (1946).

Density fluctuations are small in the near-adiabatically stratified convection zone (Gilman and Glatzmaier, 1981). Hence, the velocity correlations may be expressed as $\langle u_i u_j \rangle = \langle \rho_i \rho_j \rangle / \rho^2$ in terms of momentum density $\boldsymbol{\rho}$. We neglect the turbulence inhomogeneity when deriving effective heat conductivities and accept the simplest isotropic spectral tensor for original turbulence:

$$\langle \hat{\rho}^0_m(\mathbf{k}) \hat{\rho}^0_n(\mathbf{k}') \rangle = \delta(\mathbf{k} - \mathbf{k}') \frac{E(k)}{8\pi k^2} (\delta_{mn} - k_m k_n / k^2), \tag{3.2}$$

where $E(k)$ is the momentum density spectrum:

$$\rho^2 \langle u^2 \rangle^0 = \int_0^\infty E(k)\, \mathrm{d}k .$$

Substitution of (3.2) into (2.1) and transformation to real correlations with the use of (2.2) result in the expression

$$\langle u_i u_j \rangle = [a(\omega)\, \delta_{ij} + e_i e_j b(\omega)] \langle u^2 \rangle^0, \tag{3.3}$$

where the functions a and b of the Coriolis number $\omega = 2\tau\Omega$ are defined by (2.15). Equation (3.3) reflects the two known aspects of rotational influence on turbulence. First, an anisotropy with preferred direction $\mathbf{e} = \mathbf{\Omega}/\Omega$ is introduced. Second, the turbulence intensity is decreased: $\langle u^2 \rangle / \langle u^2 \rangle^0 = \tan^{-1}(\omega)/\omega \leq 1$. If rotation were absent, then $b = \omega = 0$, $a = \frac{1}{3}$, and (3.3) turns into the usual relation for isotropic turbulence, $\langle u_i u_j \rangle = \delta_{ij} \langle u^2 \rangle / 3$.

Substitution of (2.17) into (3.3) and then into (3.1) yields the following expressions for convective heat flux components:

$$F_r = \frac{c_p \rho \tau l^2 g}{4T} \frac{\partial \Delta T}{\partial r} \left[(a + b\cos^2\theta) \frac{\partial \Delta T}{\partial r} - b\cos\theta\sin\theta \frac{1}{r} \frac{\partial \Delta T}{\partial \theta} \right],$$

$$F_\theta = \frac{c_p \rho \tau l^2 g}{4T} \frac{\partial \Delta T}{\partial r} \left[(a + b\sin^2\theta) \frac{1}{r} \frac{\partial \Delta T}{\partial \theta} - b\cos\theta\sin\theta \frac{\partial \Delta T}{\partial r} \right]. \quad (3.4)$$

This representation takes into account the known latitude dependence and tensoral character of the effective heat conductivities for rotating fluids (Rüdiger, 1989). The fluxes (3.4) differ, however, from the usually used linear representations by explicit account for the dependence of convective heat transport coefficients on the super-adiabatic temperature gradient (see, however, Durney and Spruit, 1979). As a result, the fluxes (3.4) are nonlinear functions of ΔT.

4. Basic Assumptions, Boundary Conditions, and Numerical Method

This paper concentrates mainly on relatively deep regions of the convection zone below the supergranulation layer where rotation considerably influences convection (the surface distributions of angular velocities and temperature will be also found, however). Hydrogen and helium are nearly fully ionized at these depths. We adopt $\gamma = \frac{5}{3}$ and $c_p = 3.4 \times 10^8\ \mathrm{cm^2\,s^{-2}\,K^{-1}}$ (fully-ionized hydrogen).

The greater portion of the solar mass is concentrated below the convective envelope. We neglect self-gravitation to write $g = g_0(R/r)^2$, where R is the solar radius, and g_0 is the surface gravity.

Deviation of stratification from adiabaticity is very small ($\sim 10^{-4}$) at the depths considered (Spruit, 1974; Gough and Weiss, 1976). The adiabatic distributions will be used:

$$\rho = \rho_0 (T_a/T_0)^{1/(\gamma-1)}, \qquad P = P_0 (T_a/T_0)^{\gamma/(\gamma-1)}, \quad (4.1)$$

$$T_a = T_{\mathrm{eff}} + (g_0 R/c_p)(R/r - 1), \quad (4.2)$$

where ρ_0, P_0, and T_0 are density, pressure, and temperature at the convection zone base, and T_{eff} is the effective temperature of a star. Equations (4.1) and (4.2) do not, certainly, apply to the upper layers of the convection zone with partial ionization. The upper bound r_i for the distance r in calculations to follow is $r_i = 0.95R$. Equations (4.1) and (4.2) produce maximal errors at $r = r_i$ ($\sim 5\%$ when compared with models by Spruit, 1974, and Gough and Weiss, 1976). The errors decrease with depth ($< 1\%$ at $r = 0.7R$).

The mixing-length l is usually assumed to be proportional to pressure scale height, $l \sim -P/(\partial P/\partial r)$. The value 1.65 of the proportionality coefficient seems to fit best the observational data (Lubov, Rhodes, and Ulrich, 1980). So, we adopt

$$l = 1.65 c_p T_a (\gamma - 1)/(\gamma g), \quad (4.3)$$

where Equations (4.1) and (4.2) were used.

The next important step is to define the convective turnover time τ. We assume τ to be independent of rotational velocity. The usual estimate $\tau = l/\sqrt{\langle u^2 \rangle}$ may be applied in this case with $\langle u^2 \rangle$ evaluated from Equation (2.17) for the case of a nonrotating star. Radial heat flux obeys for this case the relation $F_r = L/4\pi r^2$, where L is stellar luminosity. We neglect radiative heat transport and adopt Equations (3.4) for heat fluxes. By taking into account that for a nonrotating fluid $\omega = 0$, and $a = \frac{1}{3}$, $b = 0$ in Equations (3.4), using the relation $\tau = l/\sqrt{\langle u^2 \rangle}$ and Equation (2.17), after some algebra we find

$$\tau = \left(\frac{16\pi r^2 c_p T_a \rho l^2}{3gL} \right)^{1/3} . \tag{4.4}$$

Convection is known to weaken with increasing angular velocity for a given superadiabaticity. At first sight, this seems to imply an increase of τ with rotation taken into account. It may be noted, however, that rotational effects should increase superadiabaticity which partly compensates for suppression of convection and the increase of τ. So, we use (4.4) as a simple approximation.

Another practice is to derive τ from linearized equations but to leave convection velocities indefinite or define them phenomenologically (Durney, 1987). However, the value of τ is controlled by nonlinear interactions of convective eddies neglected in linearized equations. We adopt in this paper an alternative approach which derives Equations (2.1) and (2.2) for convective velocities (momentum densities) but defines the intensity of original turbulence and the value of τ phenomenologically.

This paper neglects the global meridional circulation. We avoid discussing physical aspects of this assumption but simply refer to observations. Information on large-scale motions in relatively deep layers is believed to be supplied by observations of sunspot proper motions. Very slow ($\sim$2–3 m s^{-1}) circulation was found (Tuominen, Tuominen, and Kyröläinen, 1983) which varies considerably over a solar cycle and is, probably, magnetically induced (Rüdiger *et al.*, 1986).

Radiative heat transport and heating through dissipation of large-scale motions are also neglected.

Under the assumptions made above, the dynamic equations for angular velocity, Ω, and potential temperature, ΔT, read

$$\rho r \sin\theta \, \partial\Omega/\partial t = r^{-3}\, \partial(r^3 T_{r\varphi})/\partial r + r^{-1} (\sin\theta)^{-2}\, \partial(\sin^2\theta \, T_{\theta\varphi})/\partial\theta \,, \tag{4.5a}$$

$$c_p \rho \, \partial\Delta T/\partial t = -r^{-2}\, \partial(r^2 F_r)/\partial r - r^{-1} (\sin\theta)^{-1}\, \partial(\sin\theta \, F_\theta)/\partial\theta \,, \tag{4.5b}$$

where the components of the stress tensor, T_{ij}, and heat flux, $\mathbf{F}$, are defined by (2.18) and (3.4). We shall be interested mainly in the steady-state solutions of Equations (4.5).

The usual boundary conditions imposed on Equation (4.5a) are vanishing stresses, $T_{r\varphi} = 0$, at the top (R) and bottom (r_0) of the convection zone. Continuity of radial heat fluxes at these two boundaries will be accepted as the other two boundary conditions required. The constant heat flux is assumed to enter the convection zone at the lower boundary:

$$F_r = L/4\pi r_0^2 \quad \text{at} \quad r = r_0 \,.$$

The surface of a star is supposed to radiate as a black body:

$$F_r = \sigma T^4(R, \theta) \approx (L/4\pi R^2)\,(1 + 4\Delta T(R, \theta)/T_{\text{eff}}) \quad \text{at} \quad r = R \,.$$

The nonlinear equations in system (4.5) are certainly impossible to solve analytically. The problem was treated numerically. Numerical approaches meet with substantial difficulties when treating highly inhomogeneous upper layers of the convection zone. At the same time, these layers are of minor interest for the problem at hand because rotational influence on convection is small there (Coriolis numbers are small). For this reason, the surface layers were ignored in the simulations and a radius $r_i = 0.95R$ was used for the upper boundary. This, however, raises the problem of boundary conditions for $r = r_i$.

The meridional fluxes of heat, F_θ, and angular momentum, $T_{\theta\varphi}$, in the surface layer, $r_i \leq r \leq R$, are mainly caused by isotropic turbulent diffusivities, because rotationally induced effects must be small there. The diffusive smoothing-out of latitudinal inhomogeneities of angular velocity and potential temperature in the surface layer may be neglected because the layer is thin, $R - r_i \ll R$; i.e., we put $F_\theta = 0$ and $T_{\theta\varphi} = 0$ in this layer. This approximation may be understood as a neglect of latitudinal gradients in comparison to radial ones in the surface layer. In this case, the condition $T_{r\varphi} = 0$ is simply shifted to the surface $r = r_i$:

$$T_{r\varphi} = 0 \quad \text{at} \quad r = r_0 \quad \text{and} \quad r = r_i \,. \tag{4.6}$$

Reformulating the heat flux boundary condition is not so easy. We have

$$F_r(r_i, \theta)x_i^2 = (L/4\pi R^2)\,(1 + 4\Delta T(R, \theta)/T_{\text{eff}}) \,, \tag{4.7}$$

where $x_i = r_i/R$. It is necessary to exclude $\Delta T(R, \theta)$ from this expression. Note that the dependence of heat flux on latitude must be small in the surface layer: $F_r(\partial \Delta T/\partial r) \approx \text{const.}/r^2$. Therefore, $\Delta T(r_i, \theta) = \Delta T(R, \theta) + C$, where C is constant. The contribution of ΔT to the mean temperature is small and may be neglected. Note that the fluxes (3.4) and stresses (2.18) include not ΔT itself but derivatives of ΔT only. Therefore, nothing will change if we shall use $\Delta T' = \Delta T - C$ instead of ΔT, where C is an (unknown) constant difference between potential temperatures of the top and bottom of the surface layer. It is clear that $\Delta T'(r_i, \theta) = \Delta T(R, \theta)$. We shall use $\Delta T'$ instead of ΔT and omit the dash for simplicity. This leaves the absolute value of the potential temperature unknown but does not present any obstacles for determinations of spatial variations of ΔT. Equation (4.7) now reads

$$F_r(r_i, \theta) = (L/4\pi r_i^2)\,(1 + 4\Delta T(r_i, \theta)/T_{\text{eff}}) \,. \tag{4.8}$$

Note that (4.8) does not mean that the surface of a star is placed at r_i; e.g., the value of $T_a(r_i)$ as defined by (4.2) is much larger than T_{eff}.

The approximate steady solutions were found by tracing in time the relaxation of the

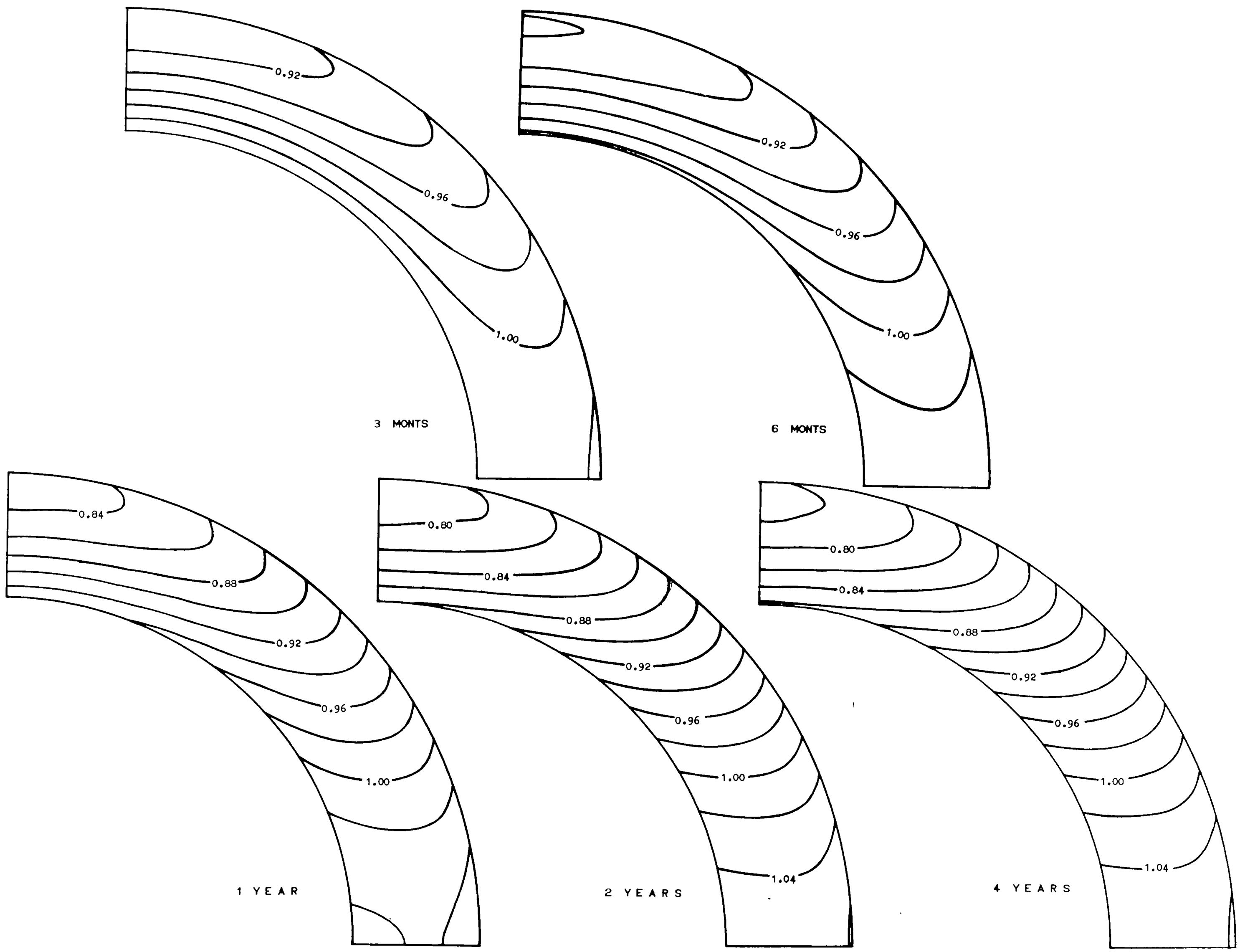

Fig. 1. Evolution of the angular velocity distribution from the starting nonequilibrium state of rigid-body rotation with angular velocity $\Omega_\odot = 2.87 \times 10^{-6}$ rad s^{-1} towards the steady state of differential rotation. The time (in real units) elapsed after the start of the run is marked near the figures. Values of normalized angular velocity $f = \Omega/\Omega_\odot$ are shown in the isoline gaps.

solutions of nonstationary equations (4.5) with boundary conditions (4.6) and (4.8) from starting (non-equilibrium) distributions of ΔT and Ω towards the steady distributions. The rigid-body rotation with a velocity Ω_0 and the distribution $\Delta T/(r)$ which is a stationary solution of Equation (4.5b) for the nonrotating case ($\omega = 0$) were used as a starting state. The time evolution of the distributions $\Omega(r, \theta, t)$ and $\Delta T(r, \theta, t)$ as governed by Equations (4.5) was followed until a moment t_0 at which the (maximal) variation rate of the distributions decreased by more than 1000 times as compared to the initial one. The functions $\Omega(r, \theta, t_0)$ and $\Delta T(r, \theta, t_0)$ were accepted as the approximate steady solutions. Such an approach was dictated partly by the planned account for mean magnetic field which, if done, will shift the major interest to dynamics.

A uniform grid over the variables $x = r/R$ and $\mu = \cos\theta$ was used with 40 grid points in each variable. An explicit finite-difference numerical scheme, conservative in both angular momentum and heat, was applied. The relative value of change of the bulk angular momentum of the convection shell due to numerical error accumulation was always smaller than 10^{-4}. The time step was approximately equal to 5×10^{-5} years, which is about four times smaller than the numerical instability threshold value.

The following values of the input parameters were adopted in the solar model: $\rho_0 = 0.2\ \mathrm{g\ cm^{-3}}$, $r_0 = 0.7R$, $g_0 = 2.74 \times 10^4\ \mathrm{cm\ s^{-2}}$, $R = 7 \times 10^{10}$ cm, $T_{\mathrm{eff}} = 5770$ K, and $\Omega_0 = \Omega_\odot = 2.87 \times 10^{-6}\ \mathrm{rad\ s^{-1}}$. Several runs were also done with Ω_0 smaller and larger than $\Omega_\odot$, but other parameters unchanged, to study the dependence of the

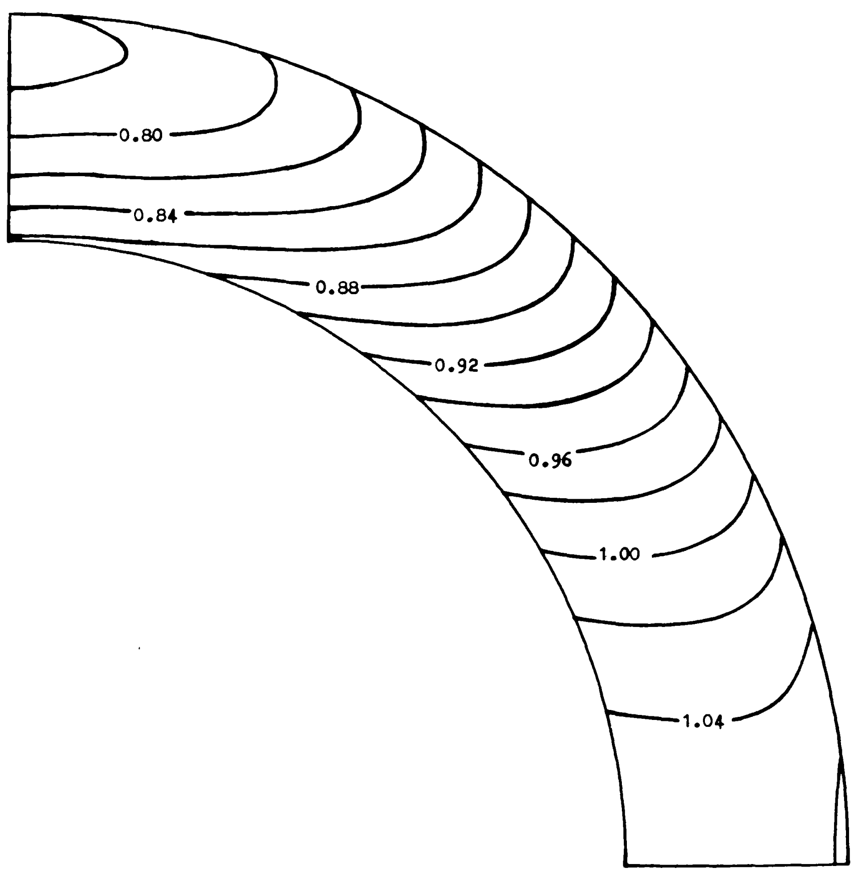

Fig. 2. Isolines of normalized angular velocity $f = \Omega/\Omega_0$ for resulting ($t \approx 5$ years) differential rotation.

simulated rotation law on the angular velocity of a Sun-like star. Next some runs were made with magnetic stresses derived in Section 2 omitted to judge the importance of fluctuating magnetic fields for the model.

4. Results and Discussion

The diffusive time, $t_d = R^2/\kappa_{\text{eff}}$, for the solar convection zone is known to be about 30 years. Surprisingly, the simulations yield an estimate of about 6 months for the time of a two-fold decrease of the deviation of ΔT and Ω from final steady distributions (we prefer using real time instead of normalized time of the numerical simulations). Figure 1 displays isorotational lines for five consecutive instants of the run. The numbers in the isoline gaps show the values of normalized angular velocity $f = \Omega/\Omega_0$. Remember that Ω_0 is the angular velocity of rigid-body rotation in the starting state. During several initial months, radial inhomogeneity of the rotation develops predominantly, with the latitudinal inhomogeneity being less pronounced. This is probably caused by the difference of radial and latitudinal scales of the convection shell. Later on, inhomogeneity of rotation in latitude reaches the value of that in the radius and then becomes larger. Note that the

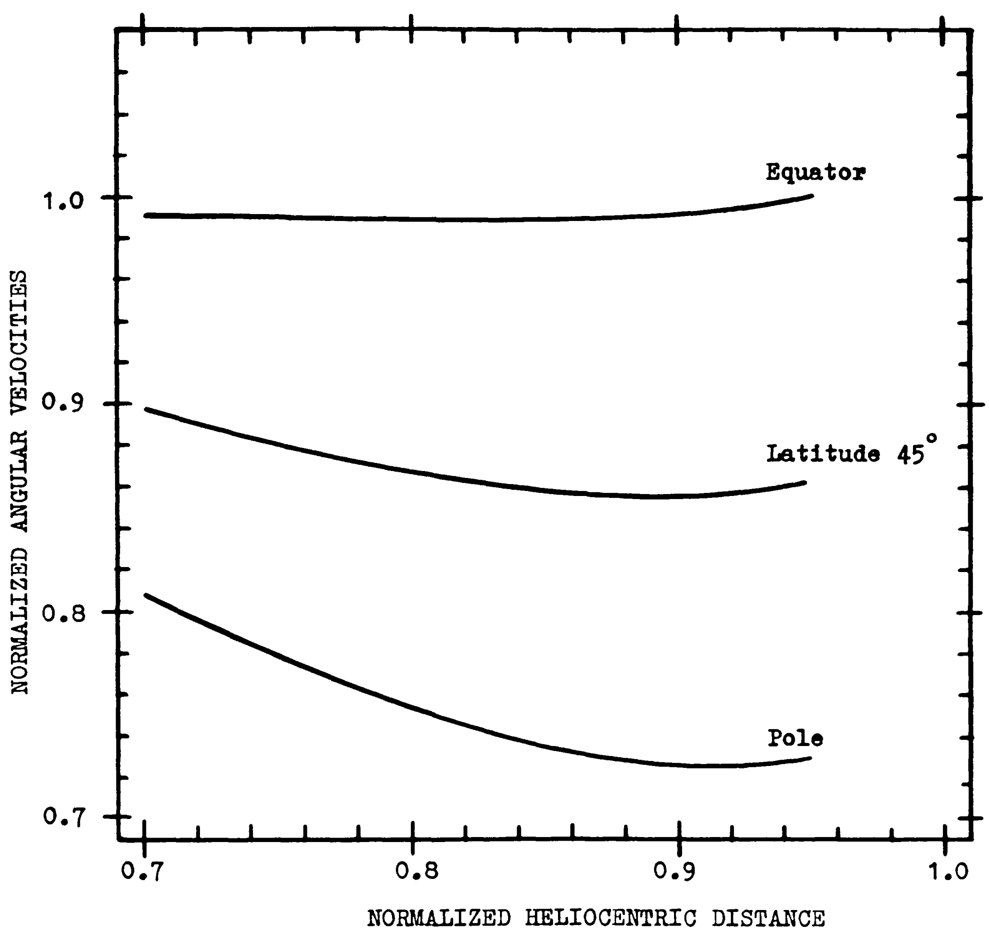

Fig. 3. Dependencies of angular velocities normalized to the equatorial value on heliocentric distance $x = r/R$ for equator, pole, and 45° latitude.

difference between angular velocity distributions for the moments of 2 and 4 years is small, indicating that the distributions are close to a steady one. The run was stopped at the instant $t_0 \approx 5$ years when the variation rate of the simulated distributions decreased by more than three orders of magnitude as compared to the initial values.

The resulting isorotational lines are shown in Figure 2. They are almost the same as shown in Figure 1 for $t = 4$ years. Helioseismology shows that variations of angular velocity with depth at low and middle latitudes are small, i.e., isorotational lines are nearly straight radial at these latitudes (Harvey, 1988; Libbrecht, 1988; Christensen-Dalsgaard and Schou, 1988; Dziembowski, Goode, and Libbrecht, 1989). Though the isolines of Figure 2 differ from radial lines, the depth-dependencies of angular velocity at low and middle latitudes are also rather weak, as is shown in Figure 3. At the same time, helioseismology (Brown, 1985; Rhodes *et al.*, 1988) and analysis of rotation of the Sun's magnetic field pattern (Stenflo, 1989, 1990) revealed a pronounced increase of angular velocity with depth at high latitudes, which can be seen in Figure 2 as well.

The depth-dependences of normalized angular velocities $\hat{\Omega} = \Omega(r, \theta)/\Omega(r_i, \pi/2)$ (the ratio of angular velocity to the surface equatorial value) for the latitudes $\lambda = 0°$, $45°$, and $90°$ are shown in Figure 3. The slow decrease of $\hat{\Omega}$ with depth at the equator is

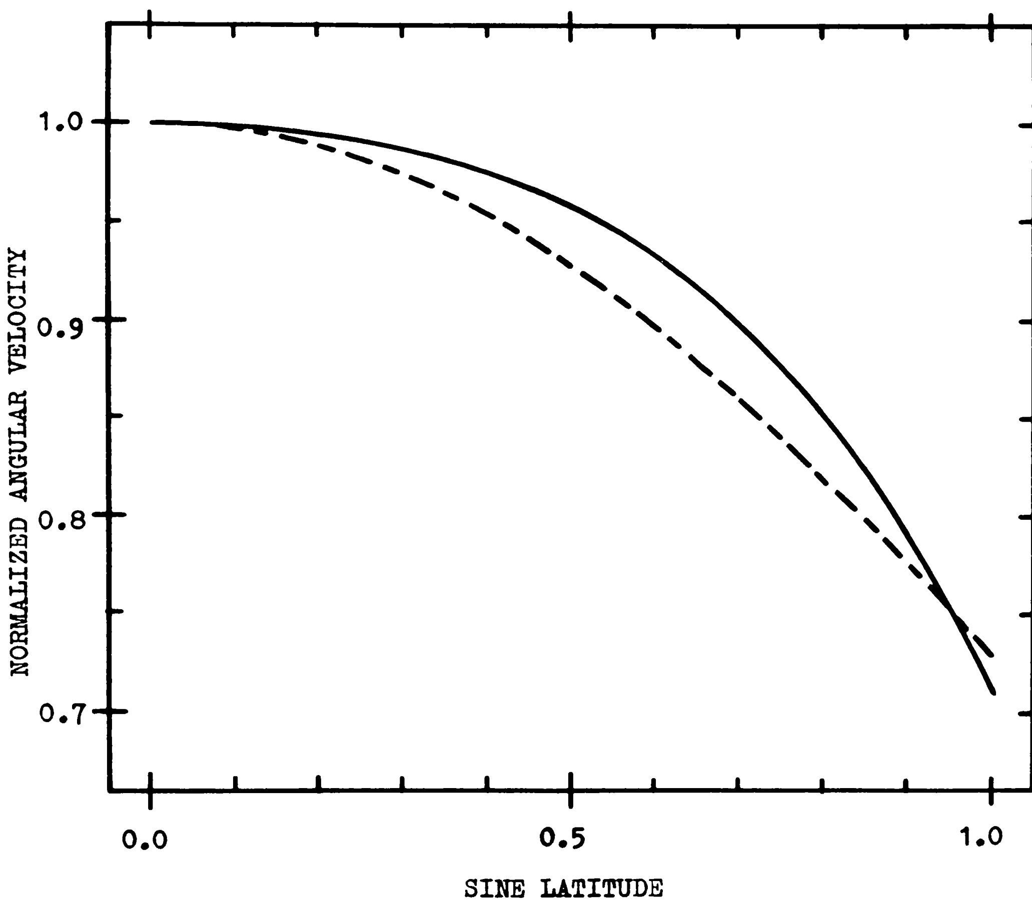

Fig. 4. Comparison of simulated differential rotation of the photosphere (broken) with Doppler measurements by Howard and Harvey (1970) (solid). Angular velocities normalized to the equatorial value are shown.

replaced by an increase at higher latitudes which is still slow at $\lambda = 45°$ but more pronounced at the pole. Figure 3 is similar (qualitatively, at least) to the depth-dependencies of Ω found by Christensen-Dalsgaard and Schou (1988) and Rhodes *et al.* (1988).

Figure 4 compares the simulated dependence of $\hat{\Omega}$ on latitude at the top boundary (broken) with the observed (Howard and Harvey, 1970) rotation of the photosphere (solid). Equator-to-pole differences of angular velocities agree rather satisfactorily. Difference in details of the latitudinal profiles are quite noticeable, however. The model somewhat overestimates the latitudinal gradients of angular velocity at low latitudes and underestimates them at high latitudes.

The simulated potential temperature, ΔT, was always small (< 10 K). The resulting equator-to-pole temperature difference is positive (pole is hotter) at all depths. This 'differential temperature' increases steadily from about 3.0 K at the bottom to about 5.1 K at the top of the convection zone. The latitude dependence of ΔT at the top is shown in Figure 5. The upper bound for the temperature difference between the solar equator and poles imposed by observations (Altrock and Canfield, 1972; Noyes, Ayres, and Hall, 1973; Falciani, Rigutti, and Roberti, 1974) is ≈ 5 K. Kuhn (1987) found some evidence favoring the value ≈ 0.6 K for the amplitude of the differential temperature.

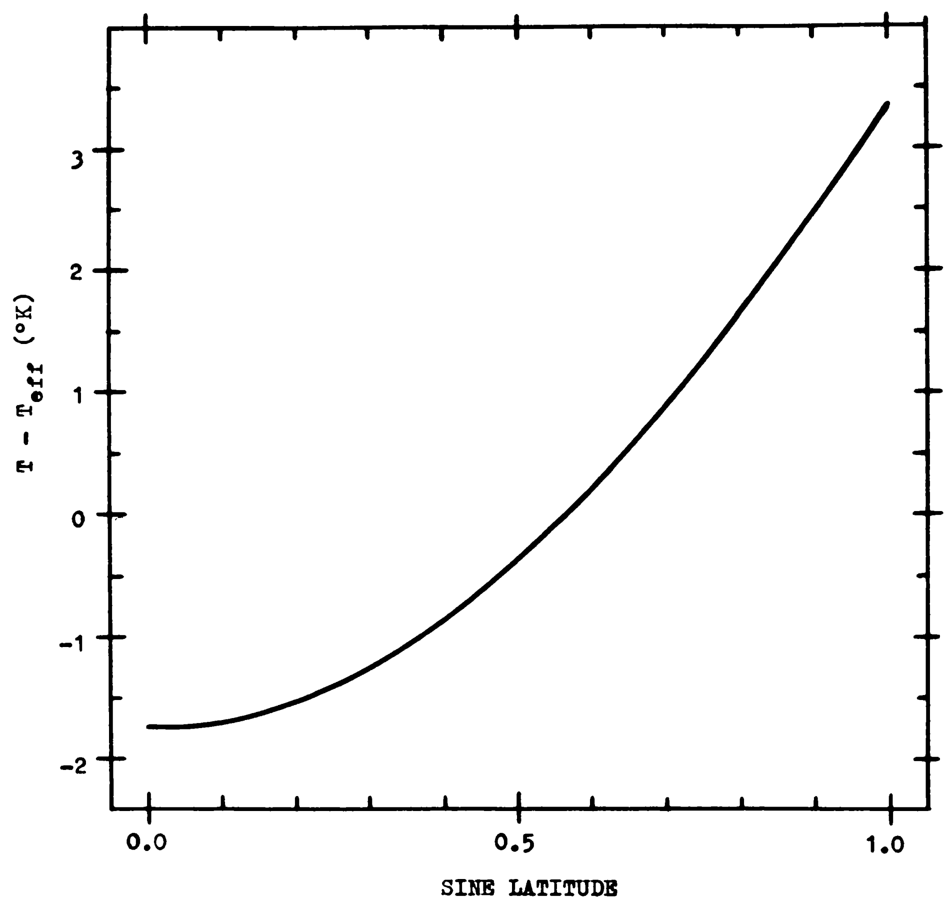

Fig. 5. The latitude variation of the simulated photospheric temperature excess $\Delta T = T - T_{\text{eff}}$.

The amplitude of 5.1 K of the model seems to be an overestimation caused, probably, by the neglect of radiative heat transport.

Let us consider again Figure 2. The shape of the isorotational lines may be interpreted as follows. The only source of rotational inhomogeneity in the model suggested is nondiffusive flux of angular momentum, λ, discussed in Section 2. Therefore, it is natural to expect that the gradient of angular velocity and (pseudo-) vector λ must be close in their directions. In other words, λ must be nearly normal to isorotational surfaces. Simple estimations show that the Coriolis number $\omega = 2\tau\Omega$ with τ defined by (4.4) is about 8 near the base of the convection zone. Hence, the flux λ in the deep regions may be estimated from the expression (2.11) valid for the rapid rotation case. This expression yields equatorward λ parallel to the axis of rotation. Therefore, the isorotational lines deep in the convection zone must be approximately normal to the axis, which is indeed the case of Figure 2. Coriolis numbers decrease with increasing r due to a decrease of τ (4.4). We have $\omega \approx 0.8$ near the top boundary. Though this value is not very small, the considerable decrease of ω means that λ should decline by an appreciable angle from the axis of rotation towards the radial direction of Equation (2.10) as r increases from r_0 to r_i. For this reason, the isorotational lines turn up when the top boundary is approached.

This interpretation is further confirmed by the results of the runs made for stars rotating faster and slower than the Sun but otherwise the same. Figure 6 shows isorotational lines for the case $\Omega_0 = 3\Omega_\odot$. In this case, Coriolis numbers are considerably greater than unity at all depths. As a result, the isolines are nearly normal to the rotation axis. Note that the relative value $\delta f = \delta\Omega/\Omega_0$ of angular velocity variation ($\delta\Omega$) over the convection zone changes little between Figures 2 and 6. This implies saturation of the

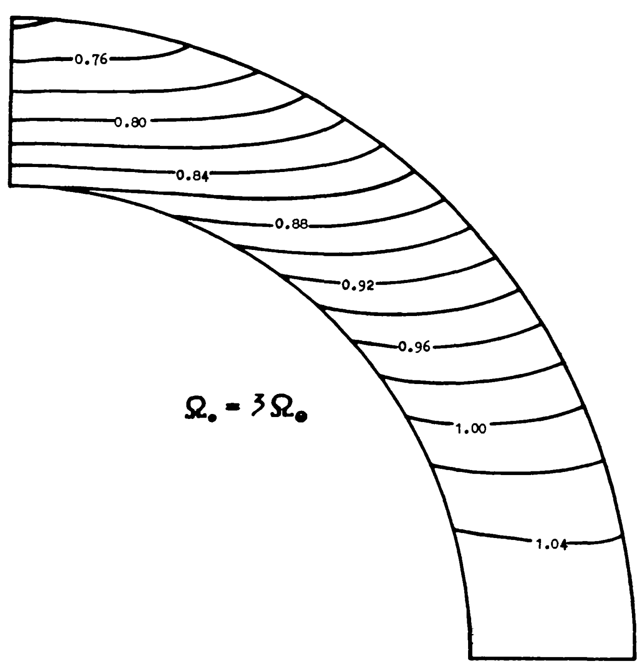

Fig. 6. Isolines of normalized angular velocity, $f = \Omega/\Omega_0$, for the 'fast rotator' with $\Omega_0 = 3\Omega_\odot$.

relative magnitude of the differential rotation for rapid rotators. The same conclusion may be drawn when examining Equation (2.11). The saturation is a distinctive feature of the mechanism for angular momentum transport by compressible convection (Paper I) adopted in the model. If the Λ-effect caused by turbulent anisotropy was used, we would find $\delta f \sim \omega^{-2}$ for rapid rotators.

Note that it is not necessary for a star to rotate faster than the Sun to be a fast rotator. Instead, the Coriolis number $\omega = 2\tau\Omega$ must be large. This condition can be satisfied by large values of τ, which often seems to be the case with K and M dwarfs. The model suggests for the fast rotators a universal rotation law with angular velocity varying by about 30% over a convection shell and isorotational lines being of the type shown in Figure 6.

The results for slow rotators are shown in Figure 7. Coriolis numbers are small and radial flux (2.10) applies to the upper layers of the convection zone in the case

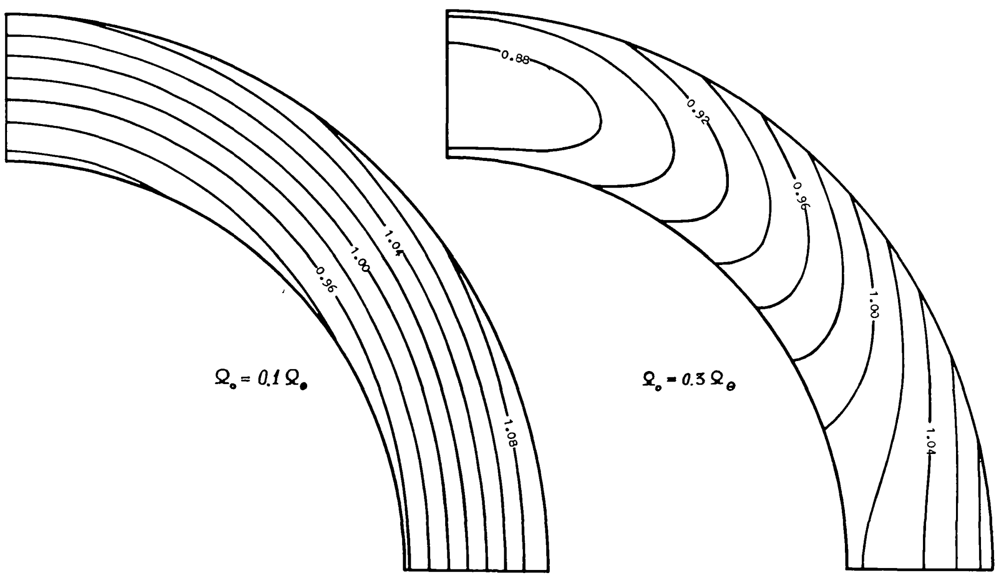

Fig. 7. Isolines of normalized angular velocity, $f = \Omega/\Omega_0$, for the 'slow rotators' with $\Omega_0/\Omega_\odot = 0.1$ and 0.3.

$\Omega_0 = 0.3\Omega_\odot$. As a result, the isolines are there nearly parallel to the top boundary. Near the bottom, however, the parameter ω is still greater than unity and the isorotational lines change towards the direction normal to the rotation axis. An exception is the region close to the equatorial plane where the flux λ is nearly radial at all values of ω. In the case $\Omega_0 = 0.1\Omega_\odot$, the values of ω are smaller than unity at all depths. For this reason, Equation (2.10) applies and variations of angular velocity with latitude are small. The runs with still smaller values of Ω_0 show that the resulting distribution of angular velocity approaches spherical symmetry in the slow rotation limit. The relative value of rotational inhomogeneity, $\delta f = (\Omega_{top} - \Omega_{bottom})/\Omega_{top}$, tends to a constant $\delta f = 0.18$ as $\Omega_0/\Omega_\odot$ progressively decreases.

Hence, the asymptotic rotation laws with constant δf are found for both very slow and very fast rotators.

Several runs were made with Maxwell stresses of Section 2 omitted to judge the importance of magnetic contributions for the model. The neglect of magnetic stresses changes the results drastically. In the slow rotation case ($\Omega_0 = 10^{-3}\,\Omega_\odot$) the resulting distribution of angular velocity was also spherically symmetric but rotational inhomogeneity increased to the value of $\delta f = 0.579$ as compared to $\delta f = 0.180$ for the above model. The ratio of δf for nonmagnetic case to that for magnetic one increases further with Ω_0 until the nonmagnetic system becomes unstable at $\Omega_0 \geq \Omega_t$ (the threshold value, Ω_t, was not exactly determined but is close to $0.27\Omega_\odot$). In the instability region, the 'nonmagnetic' runs show initially linear and than exponential growth of rotational inhomogeneity with time with no signs of relaxation. Therefore, the Maxwell stresses are rather important for the model developed.

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