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MAGNETOSPHERIC PLASMA

Polarization Splitting of the Alfvén Wave Spectrum in a Dipole Magnetosphere with a Rotating Plasma

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Abstract—Alfvén waves in a dipole magnetosphere with a rotating plasma are studied theoretically. The plasma-motion-related properties of azimuthally small-scale standing Alfvén waves having nearly poloidal or nearly toroidal polarization are analyzed. Equations are obtained that describe the longitudinal (along the magnetic field) structure and spectra of the waves having such polarizations. The equations obtained are then solved both analytically (in the Wentzel–Kramers–Brillouin approximation) and numerically. Attention is focused on the polarization splitting of the spectrum—the difference between the eigenfrequencies of the toroidally and poloidally polarized Alfvén waves. The distribution of this difference in a direction across the magnetic shells is analyzed. It is shown that, unlike in the models in which the plasma is assumed to be at rest, taking into account rotation of the magnetosphere plasma results in an additional splitting of the spectrum of the poloidal Alfvén waves due to the difference in their azimuthal mode numbers.

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1. INTRODUCTION

Modern theoretical research on MHD oscillations in the Earth's magnetosphere is generally thought to begin with a paper by Dungey [1], who derived equations describing the structure and spectrum of poloidal and toroidal Alfvén waves by using a magnetosphere model with a dipole magnetic field. In an axisymmetric model such as this one, arbitrary oscillations can be described by a superposition of azimuthal modes of the form of $\exp(im\varphi)$, where $m = 0, 1, 2, 3, \ldots$ is the azimuthal mode number and φ is the azimuthal angle. Oscillations with $m = 0$ are called toroidally polarized, while oscillations with $m \longrightarrow \infty$ are referred to as poloidally polarized. In toroidal Alfvén waves, the magnetic field and the plasma oscillate in the azimuthal direction and the electric field oscillates across the magnetic shells. And vice versa, in poloidal Alfvén waves, the magnetic field and the plasma oscillate across the magnetic shells and the electric field executes azimuthal oscillations.

A theory of Alfvén waves with $m \gg 1$ in a magnetosphere with a dipole-like magnetic field was worked out in [2]. Structurally, such oscillations along the magnetic field lines are standing Alfvén waves between the magneto-conjugated ionospheres. In the transverse direction, these oscillations are localized between two resonant magnetic shells. In the vicinity of one of these shells, the radial (transverse) wavelength of the oscillations is much less than their azimuthal wavelength and the oscillations have almost poloidal polarization. In [2], this shell was called the poloidal resonant surface. It is near this shell that a monochromatic source (e.g., external currents in the Earth's ionosphere) can generate the oscillations under study.

These oscillations propagate across the magnetic shells toward another (toroidal) resonant surface. During propagation, their polarization changes from nearly poloidal to nearly toroidal. In the vicinity of the toroidal resonant surface, Alfvén waves with $m \gg 1$ are almost completely absorbed due to the Joule dissipation of their energy in the ionosphere. Standing Alfvén waves with toroidal polarization have somewhat different eigenfrequencies than those with poloidal polarization. The difference between these eigenfrequencies is called the polarization splitting of the spectrum. Knowing this difference, one can determine the distance between the toroidal and the poloidal resonant magnetic shells.

The amplitude of the oscillations between the poloidal and the toroidal resonant surfaces is governed by two competing effects—the dissipation of oscillations in the ionosphere and their possible enhancement by high-energy charged particle beams. It is known [3, 4] that Alfvén waves can grow in their interaction with high-energy particles present in the plasma. If the dissipation of Alfvén waves in the ionosphere predominates over their enhancement (the damping rate is larger than the growth rate), their amplitude decreases from the poloidal resonant surface toward the toroidal one. The only exception may be in a narrow vicinity of the toroidal resonant surface, where the singularity in the field of oscillations is regularized by their damping. In the opposite case (when the growth rate is larger than the damping rate), the oscillation amplitude increases

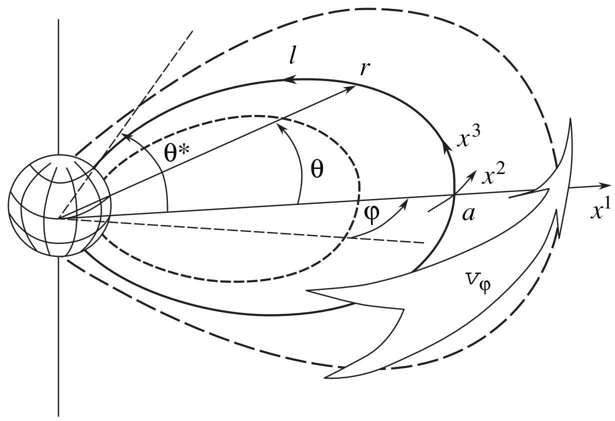

Fig. 1. Geometry of a magnetosphere model with a dipole magnetic field and with a plasma rotating in the azimuthal direction, $\mathbf{v} = (0, v_\varphi, 0)$. The coordinate systems associated with the magnetic field lines are shown: the curvilinear orthogonal coordinate system (x^1, x^2, x^3) and the nonorthogonal coordinate system (a, φ, θ) used in numerical simulations.

monotonically from the poloidal toward the toroidal resonant surface.

In all earlier papers on the subject, the polarization splitting of the spectrum of standing Alfvén waves was determined by using magnetosphere models in which the plasma was assumed to be immobile. However, the actual magnetosphere is a dynamically equilibrium plasma configuration. In other words, motion is an inherent property of the magnetospheric plasma and, of course, plays an important role in the formation of the structure and spectrum of Alfvén waves in the magnetosphere. In the present paper, we make a first attempt to investigate how the motion of the magnetospheric plasma influences the longitudinal structure and spectrum of standing Alfvén waves with $m \gg 1$ near the poloidal and toroidal resonant surfaces.

Our paper is organized as follows. In Section 2, we briefly describe the magnetosphere model and derive basic equations for the longitudinal (along the magnetic field) structure and spectrum of nearly toroidal and nearly poloidal standing Alfvén waves with $m \gg 1$ in the vicinities of the corresponding resonant surfaces. In Section 3, the equation for standing Alfvén waves near the toroidal resonant surface is solved in the Wentzel–Kramers–Brillouin (WKB) approximation. In Section 4, we perform a similar analysis for standing Alfvén waves near the poloidal resonant surface and obtain an exact expression for the polarization splitting of the Alfvén wave spectrum. In Section 5, we derive analytic expressions for the polarization splitting of the waves under consideration and determine the characteristic spatial scale on which they are localized across the magnetic shells. In Section 6, the results of numerical calculations carried out for the first several modes of the standing Alfvén waves are presented and discussed. Finally, in the Conclusions, we summarize the main results of our work.

2. MODEL OF THE MEDIUM AND BASIC EQUATIONS

In order to take into account the motion of the medium, we use a magnetosphere model with a dipole magnetic field and with a plasma rotating in the azimuthal direction. The geometry of the model under analysis is shown in Fig. 1. We introduce an orthogonal curvilinear coordinate system (x^1, x^2, x^3) associated with the magnetic field lines (see Fig. 1). The x^3 coordinate is directed along a field line, the x^1 coordinate is orthogonal to the magnetic shells, and the x^2 coordinate points in the azimuthal direction so as to complete a right-handed system. In these coordinates, $\mathbf{B}_0 = (0, 0, \hat{B}_{03})$ is the unperturbed magnetic field vector, $\mathbf{v}_0 = (0, \hat{v}_{02}, 0)$ is the unperturbed plasma velocity vector (the notation $\hat{B}_{03}$ and $\hat{v}_{02}$ corresponds to the physical components of the vectors), and the square of the length element has the form

$$ds^2 = g_1(dx^1)^2 + g_2(dx^2)^2 + g_3(dx^3)^2,$$

where g_i $(i = 1, 2, 3)$ are the metric tensor components. If the role of the azimuthal coordinate x^2 is played by the azimuthal angle φ, we have $\hat{v}_{02} \equiv v_\varphi = \sqrt{g_2}\Omega$, where Ω is the angular velocity of the plasma.

A detailed analytic self-consistent model of such a magnetosphere was presented in [5]. A rotating plasma configuration is maintained in equilibrium by the gas-kinetic plasma pressure gradient. The direction of the plasma rotation is chosen to agree with the symmetry of the problem. The plasma medium is assumed to be homogeneous in the azimuthal direction, and the azimuthal plasma motion is such that it does not influence the geometry of the magnetic shells. This is why it is possible to choose an arbitrary axisymmetric magnetic field configuration, e.g., a dipole configuration. It is the dipole magnetic field component that dominates the inner part of the Earth's magnetosphere. It can be shown [5] that, in this case, the angular plasma velocity is constant along each magnetic shell and is a function of the transverse coordinate solely, $\Omega \equiv \Omega(x^1)$. Since the problem is symmetric with respect to the azimuthal coordinate x^2, the metric tensor components depend only on the transverse, x^1, and longitudinal, x^3, coordinates: $g_i \equiv g_i(x^1, x^3)$.

Note that an equilibrium magnetosphere model with a rotating plasma can be constructed in other ways. For instance, in [6, 7], axisymmetric models were considered in which the plasma was assumed to be held in equilibrium by the Earth's gravitational field and by the self-consistent generation of currents in a differentially rotating plasmasphere. In those two papers, it was shown that, with such axisymmetric models, it is possible to explain the main characteristic features of the generation of electric fields and currents in the near-Earth plasma and in the Earth's ionosphere. The models also make it possible to calculate the distribution of magnetospheric plasma in the meridional plane. Our further analysis, however, will be carried out with a simpler model developed in [5] because it makes the calculations easier. In particular, this model yields a rather simple analytic expression for the Alfvén velocity distribution $A(x^1, x^3)$ in the magnetic meridian plane. A particular expression for the distribution A will be given in Section 5, where we will present the results of numerical simulations.

In the magnetosphere model used here, we describe Alfvén waves by the set of ideal MHD equations

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla P + \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B}, \quad (1a)$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}), \quad (1b)$$

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{v}) = 0, \quad (1c)$$

$$\frac{d}{dt}\frac{P}{\rho^\gamma} = 0, \quad (1d)$$

where $\mathbf{B}$ and $\mathbf{v}$ are the magnetic field and plasma velocity vectors, P and ρ are the plasma pressure and plasma mass density, and γ is the adiabatic index. In Eqs. (1a) and (1d), $d/dt = \partial/\partial t + \mathbf{v}\cdot\boldsymbol{\nabla}$ is the time derivative in the frame of reference of a moving plasma element. The time-independent set of Eqs. (1) describes the distributions of the equilibrium parameters $\mathbf{B}_0$, $\mathbf{v}_0$, P_0, and ρ_0 in a steady state (see [5]). The parameters perturbed by Alfvén waves will be denoted by $\tilde{\mathbf{B}}$, $\tilde{\mathbf{v}}$, $\tilde{P}$, and $\tilde{\rho}$.

Let us consider monochromatic waves of the form $\exp(-i\omega t + ik_2x^2)$, where ω is the wave frequency and k_2 is the azimuthal wavenumber (for $x^2 = \varphi$, we have $k_2 \equiv m = 0, 1, 2, \ldots$). We linearize Eqs. (1) in the small perturbations introduced by Alfvén waves. From Eqs. (1a) and (1b) we obtain

$$-\rho_0(i\bar{\omega}v_1 + v_2\Omega\nabla_1\ln g_2) - \frac{\tilde{\rho}\Omega^2}{2}\nabla_1 g_2 = -\nabla_1\tilde{P} + \frac{B_0}{4\pi}\frac{1}{\sqrt{g_3}}(\nabla_3B_1 - \nabla_1B_3), \quad (2a)$$

$$\rho_0\left(-i\bar{\omega}v_2 + v_1\frac{\nabla_1(g_2\Omega)}{g_1} + \frac{v_3\Omega}{g_3}\nabla_3 g_2\right) = -ik_2\tilde{P} + \frac{B_0}{4\pi}\frac{1}{\sqrt{g_3}}(ik_2B_3 - \nabla_3B_2), \quad (2b)$$

where $\nabla_i \equiv \partial/\partial x^i$ ($i = 1, 2, 3$), v_i and B_i are the covariant components of the vectors $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{B}}$, $\bar{\omega} = \omega - k_2v_0^2$ is the wave frequency in a moving plasma with allowance for the Doppler effect, and v_0^2 is the contravariant component of the unperturbed plasma velocity (for $k_2 = m$, we have $v_0^2 = \Omega$).

It is convenient to investigate Alfvén waves by expressing the components of their electromagnetic and velocity fields in terms of potentials. According to the Helmholtz decomposition theorem [8], any differentiable vector field can be represented as the sum of an irrotational and a solenoidal field. We represent the perturbed electric field as

$$\tilde{\mathbf{E}} = -\nabla\Phi + \nabla\times\boldsymbol{\Psi},$$

where Φ and $\boldsymbol{\Psi} = (\psi_1, \psi_2, \psi_3)$ are the scalar and vector potentials, respectively. Obviously, the field $\tilde{\mathbf{E}}$ is invariant with respect to adding an arbitrary constant to Φ, so, without loss of generality, the constant can be set equal to zero. The field $\tilde{\mathbf{E}}$ is also invariant with respect to adding the gradient of an arbitrary function to the vector potential, $\boldsymbol{\Psi} + \nabla\chi$. Choosing the function χ to satisfy the equation $\psi_1 + \nabla_1\chi = 0$, we can write

$$\boldsymbol{\Psi} = (0, \xi, \psi),$$

where $\xi = \psi_2 + \nabla_2\chi$ and $\psi = \psi_3 + \nabla_3\chi$. In the linear approximation, the components of the electric field $\tilde{\mathbf{E}}$ are related to the components of the magnetic field $\tilde{\mathbf{B}}$ and velocity field $\tilde{\mathbf{v}}$ by the drift equation

$$\tilde{\mathbf{E}} = \frac{1}{c}(\tilde{\mathbf{v}}\times\mathbf{B}_0 + \mathbf{v}_0\times\tilde{\mathbf{B}}).$$

From this equation, together with Eq. (1b), we can express the components of $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{v}}$ through the potentials Φ, ξ, and ψ. Using Eqs. (1c) and (1d), we can also express the perturbed plasma density and pressure, $\tilde{\rho}$ and $\tilde{P}$, in terms of these potentials.

Deriving equations for the total field of MHD oscillations, including both Alfvén and magnetosonic waves, is a rather complicated procedure. In what follows, we will be interested in the spectrum and in the longitudinal (along the magnetic field lines) structure of standing Alfvén waves with $m \gg 1$ near the poloidal and toroidal resonant surfaces. In the absence of dissipation or other scattering mechanisms, there is a singularity in the amplitude distribution of Alfvén waves on the toroidal resonant magnetic surfaces, at which the source frequency coincides with the local Alfvén frequency. In this case, the potentials Φ, ξ, and ψ have different types of singularities associated with the Alfvén wave. The scalar potential Φ has the highest order singularity [9–11]. The problem of the structure of Alfvén waves near resonant surfaces can be formulated by retaining only the highest order terms in the equations.

In the leading order of the perturbation theory, the field components of the resonant Alfvén waves can be expressed in terms of the potential Φ:

$$E_1 = -\nabla_1\Phi, \quad E_2 = -ik_2\Phi, \quad E_3 = \frac{k_2\Omega}{\bar{\omega}}\nabla_3\Phi,$$

$$B_1 = \frac{k_2 c}{\bar{\omega}}\frac{g_1}{\sqrt{g}}\nabla_3\Phi,$$

$$B_2 = i\frac{c}{\bar{\omega}}\frac{g_2}{\sqrt{g}}\nabla_3\left(\nabla_\perp - k_2\frac{(\nabla_1\Omega)}{\bar{\omega}}\right)\Phi, \; B_3 = 0, \tag{3}$$

$$v_1 = -i\frac{k_2}{p}\frac{c}{B_0}\Phi, \quad v_2 = \frac{cp}{B_0}\nabla_1\Phi,$$

$$v_3 \approx i\frac{cp}{B_0}\frac{\Omega}{\bar{\omega}}(\nabla_3 \ln g_2)\nabla_1\Phi,$$

where we have introduced the notation $g = g_1g_2g_3$ and $p = \sqrt{g_2/g_1}$. In deriving the expression for the velocity component v_3, perturbed density $\tilde{\rho}$, and perturbed pressure $\tilde{P}$, we used the smallness of two parameters, namely, the ratios S/A and $\hat{v}_{02}/A \ll 1$, where $S = \sqrt{\gamma P_0/\rho_0}$ is the speed of sound and $A = B_0/\sqrt{4\pi\rho_0}$ is the Alfvén velocity in the plasma. For the Earth's magnetosphere, the characteristic values are $\hat{v}_{02}$, $S \le 50$ km/s and $A \sim 10^3$ km/s.

The equation for Alfvén waves is derived by acting by the operator $\nabla_1 B_0/\rho_0$ upon Eq. (2b) from the left and by taking the difference between the resulting equation and Eq. (2a) multiplied by ik_2B_0/ρ_0. For a region just around the toroidal resonant surface (where $|\nabla_1\Phi/\Phi| \gg m \gg 1$), we retain only the leading-order terms, proportional to $\sim\nabla_1^2\Phi$, to arrive at the following longitudinal equation for the toroidal Alfvén waves:

$$\hat{L}_T\Phi = 0. \tag{4}$$

Here,

$$\hat{L}_T = \hat{L}_{T0} - \beta_T \tag{5}$$

is the toroidal longitudinal operator, in which

$$\hat{L}_{T0} = \frac{1}{\sqrt{g_3}}\nabla_3\frac{p}{\sqrt{g_3}}\nabla_3 + p\frac{\bar{\omega}^2}{A^2}$$

is a zero-order toroidal operator and

$$\beta_T = p\frac{\Omega^2}{A^2}\left(\frac{\nabla_3 g_2}{\sqrt{g_2g_3}}\right)^2$$

is a small correction introduced by the rotation of the plasma and by its finite pressure.

For a region just around the poloidal resonant surface (where $|\nabla_1\Phi/\Phi| \ll m$), we retain only the leading-order terms, proportional to $\sim m^2$, in the equation obtained and arrive at the following longitudinal equation for the nearly poloidal Alfvén waves:

$$\hat{L}_P\Phi = 0. \tag{6}$$

Here,

$$\hat{L}_P = \hat{L}_{P0} - \beta_P \tag{7}$$

is the poloidal longitudinal operator, in which

$$\hat{L}_{P0} = \frac{1}{\sqrt{g_3}}\nabla_3\frac{p^{-1}}{\sqrt{g_3}}\nabla_3 + p^{-1}\frac{\bar{\omega}^2}{A^2}$$

is a zero-order poloidal operator and

$$\beta_P = p\frac{\Omega^2}{2A^2}(\nabla_1 \ln g_2)\left(\nabla_1 \ln\frac{\rho_0\sqrt{g_3}}{B_0}\right) + \frac{S^2}{A^2}\frac{\nabla_1 \ln\bar{\omega}/B_0}{\sqrt{g_1g_2}}\left(\nabla_1 \ln\frac{P_0^{1/\gamma}\sqrt{g_3}}{B_0}\right)$$

is a correction introduced by the rotation of the plasma and by its finite pressure. We restrict ourselves to considering only such values $m \gg 1$ for which the term β_P in the operator $\hat{L}_P$ makes a small correction to the zero-order operator.

3. STRUCTURE AND SPECTRUM OF THE TOROIDAL ALFVÉN WAVES IN THE MAGNETOSPHERE

Equation (4) for the toroidal Alfvén waves can be rewritten as

$$\frac{\partial}{\partial l}p\frac{\partial}{\partial l}\Phi + \left(p\frac{\bar{\omega}^2}{A^2} - \beta_T\right)\Phi = 0, \tag{8}$$

where $dl = \sqrt{g_3}dx^3$ is an element of length along the magnetic field line. This equation should be supplemented with the boundary conditions at the ionosphere. These conditions play an important role in the generation of Alfvén waves and in their absorption. Thus, external currents can generate standing Alfvén waves with $m \gg 1$ in the ionosphere, which are then dissipated due to the finite plasma conductivity [12]. However, in solving the problem of the spectrum and longitudinal structure of the main modes of the standing Alfvén waves under examination here, there is no need to account for the finite conductivity of the ionospheric plasma. In order to describe such large-scale low-frequency oscillations, it is sufficient to use an approximation in which the ionosphere is modeled as a perfectly conducting spherical shell. The finite conductivity of the ionospheric plasma gives rise to small corrections to the longitudinal structure of the main modes and to the spectrum of their eigenfrequencies. In the subsequent analysis, these corrections will be ignored. Considering the ionosphere as a perfectly conducting boundary, we set $\mathbf{E}_\perp = -\nabla_\perp \Phi = 0$ at it to obtain the boundary conditions for the potential Φ:

$$\Phi(l_\pm) = 0, \quad (9)$$

where $l_\pm$ are the footpoints of a magnetic field line in the ionospheres of the northern (plus sign) and southern (minus sign) hemispheres. Equation (8) with boundary conditions (9) constitutes an eigenvalue problem for toroidal Alfvén waves.

In order to qualitatively understand the structure and spectrum of these waves, we solve the problem given by Eq. (8) with boundary conditions (9) in the WKB approximation with respect to the l coordinate. The WKB approximation is valid for waves with wavelengths much shorter than the characteristic scale of the plasma inhomogeneity in the magnetosphere. We seek a solution to Eq. (8) in the form

$$\Phi = \exp(is(l)),$$

where $s(l)$ is the large quasiclassical phase, which will be represented as the series $s = s_0 + s_1 + s_2 + \dots$ ($|s_0| \gg |s_1| \gg |s_2| \dots$). We assume that β_T is a second-order correction in perturbation theory. In the zeroth and first orders of the WKB approximation, we then obtain from Eq. (8) the relationships

$$s_0 = \pm\bar{\omega}\int dl'/A, \quad s_1 = \frac{i}{2}\ln(p/A).$$

The solution to Eq. (8) that satisfies boundary conditions (9) has the form

$$\Phi_{TN} = C_{TN}\sqrt{\frac{A}{p}}\sin\left(\Omega_N \int_{l_-}^{l} \frac{dl'}{A}\right). \quad (10)$$

Here, C_{TN} is an arbitrary constant; $\Omega_N = \pi N/t_A$, ($N = 1, 2, \dots$) is the wave frequency, which plays the role of the eigenvalue in the problem given by Eq. (8) with boundary conditions (9); and

$$t_A = \int_{l_-}^{l_+} \frac{dl'}{A}$$

is the time required for a wave to propagate with the Alfvén velocity along a magnetic field line between the magneto-conjugated ionospheres. For each N value, solution (10) describes a standing Alfvén mode with $N - 1$ nodes along a magnetic field line. In the second order of the WKB approximation, we obtain the following correction to the quasiclassical phase:

$$s_2 = \frac{1}{8\bar{\omega}}\int A\left\{(\ln A)'^2 - (\ln p)'^2 + 2(\ln A)'' - 2(\ln p)'' - 4\frac{\beta_T}{p}\right\}dl'.$$

The general expression for the frequency of toroidal Alfvén eigenmodes can then be written as

$$\Omega_{TN} = \Omega_N - \frac{1}{8\pi N}\int_{l_-}^{l_+} A\left\{(\ln A)'^2 - (\ln p)'^2 + 2(\ln A)'' - 2(\ln p)'' - 4\frac{\beta_T}{p}\right\}dl'. \quad (11)$$

The solution obtained is applicable only to modes with $N \gg 1$. For the main modes, whose wavelengths are comparable to the characteristic scale of the plasma inhomogeneity, the problem given by Eq. (8) with boundary conditions (9) can be solved numerically. The wave structure of the first three modes is shown in Fig. 2, and the dependence of their frequencies on the position of the magnetic shell in the magnetosphere is illustrated in Figs. 3 and 4. The results of numerical simulations will be discussed in more detail in Section 6.

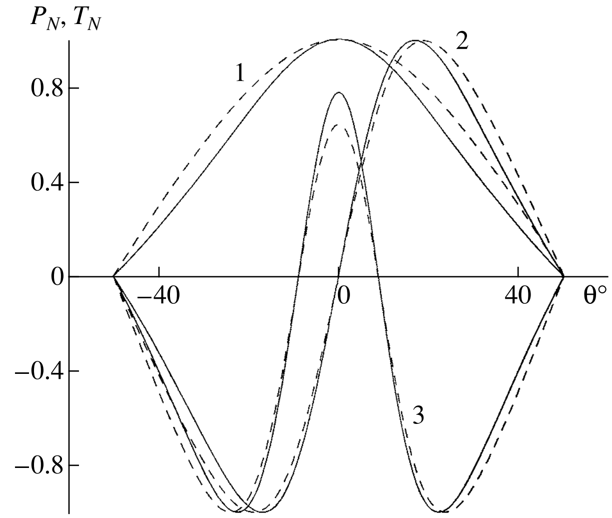

Fig. 2. Structures of standing Alfvén waves with toroidal (solid curves) and with poloidal (dashed curves) polarization: the poloidal, P_N, and toroidal, T_N, eigenfunctions of unit amplitude for the first three longitudinal modes (N = 1, 2, 3).

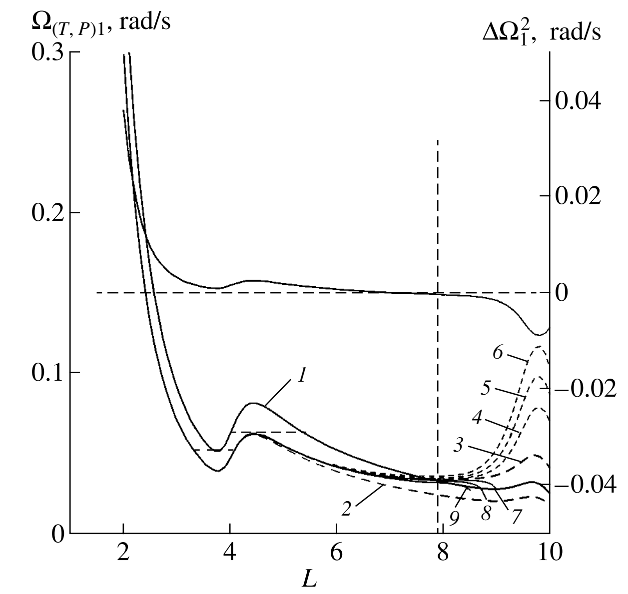

Fig. 3. Eigenfrequencies $\Omega_{(T,P)1}$ of the fundamental longitudinal modes of toroidal (curve *1*) and poloidal (curves *2*–*9*) Alfvén waves vs. magnetic shell parameter L (the family of the lower curves, referring to the left ordinate). The numerals *2*–*9* correspond to the poloidal eigenfrequencies (*2*) in a cold plasma at rest ($P_0 = 0$, $\Omega = 0$), (*3*) in a plasma with a finite gas-kinetic pressure ($P_0 \neq 0$) and with $\Omega = 0$, and in a finite-pressure rotating plasma ($P_0 \neq 0$, $\Omega \neq 0$) for different azimuthal mode numbers m = (*4*) –20, (*5*) –50, (*6*) –100, (*7*) 20, (*8*) 50, and (*9*) 100. The upper curve, referring to the right ordinate, shows the spectral splitting of the eigenfrequencies, $\Delta\Omega_1^2 = \Omega_{T1}^2 - \Omega_{P1}^2$, vs. magnetic shell parameter L for $m = -50$.

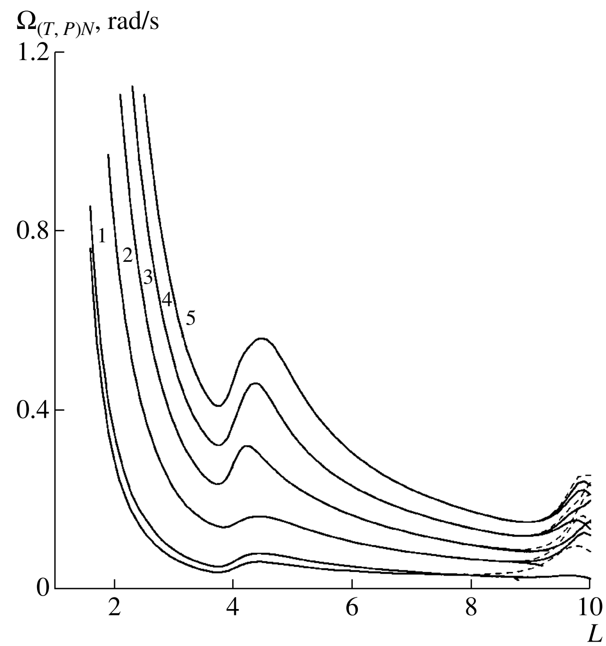

Fig. 4. Eigenfrequencies $\Omega_{(T,P)N}$ of the first five modes ($N = 1, \ldots, 5$) of toroidal (heavy curves) and poloidal (solid and dashed light curves) Alfvén waves with the azimuthal mode numbers $m = 50$ (solid curves) and $m = -50$ (dashed curves) as functions of the magnetic shell parameter L.

4. STRUCTURE AND SPECTRUM OF THE POLOIDAL ALFVÉN WAVES IN THE MAGNETOSPHERE

Equation (6), which describes the structure of poloidal Alfvén waves, can be represented as

$$\frac{\partial}{\partial l}p^{-1}\frac{\partial}{\partial l}\Phi + \left(p^{-1}\frac{\bar{\omega}^2}{A^2} - \beta_P\right)\Phi = 0. \quad (12)$$

For poloidal Alfvén waves, the boundary conditions at the perfectly conducting ionosphere has the same form as boundary conditions (9) for the toroidal waves. In the zeroth and first orders of the WKB approximation, the solution to Eq. (12) with boundary conditions (9) has the form

$$\Phi_{PN} = C_{PN}\sqrt{Ap}\sin\left(\Omega_N\int_{l_-}^{l}\frac{dl'}{A}\right). \quad (13)$$

Solution (13) differs from solution (10) for toroidal Alfvén waves only in the preexponential factor.

In the second order of the WKB approximation, the eigenfrequencies of the poloidal Alfvén waves are given by the expression

$$\Omega_{PN} = \Omega_N - \frac{1}{8\pi N}\int_{l_-}^{l_+}A\{(\ln A)'^2 - (\ln p)'^2 + 2(\ln A)'' - 2(\ln p)'' - 4\beta_P p\}dl', \quad (14)$$

which differs from the analogous expression (11) for the eigenfrequencies of the toroidal waves only in the corresponding two terms in the integrand.

5. LOCALIZATION OF STANDING ALFVÉN WAVES WITH $m \gg 1$ ACROSS THE MAGNETIC SHELLS

The quantity $\Delta\Omega_N = \Omega_{TN} - \Omega_{PN}$, which is called the polarization splitting of the spectrum, plays an important role in determining the structure of the standing Alfvén waves with $m \gg 1$ and their localization across the magnetic shells [2]. In [13], the following model equation was derived in order to describe the wave structures across the shells:

$$\nabla_1(\omega^2 - \Omega_{TN}^2)\nabla_1 U - k_y^2(\omega^2 - \Omega_{PN}^2)U = 0,$$

where $k_y \sim m$ is the azimuthal wavenumber. In the WKB approximation with respect to the x^1 coordinate, one can obtain the following expression for the quasiclassical wavenumber:

$$k_1^2 = k_y^2\frac{\omega^2 - \Omega_{PN}^2}{\Omega_{TN}^2 - \omega^2}.$$

For most of the magnetosphere, the functions $\Omega_{PN}(x^1)$ and $\Omega_{TN}(x^1)$ decrease monotonically with increasing coordinate x^1 (see Fig. 4). In the vicinity of closely spaced resonant surfaces $x^1 = x_{PN}^1$ and $x^1 = x_{TN}^1$, these functions can be approximated by the linear expressions

$$\Omega_{PN}(x^1) \approx \omega\left(1 - \frac{x^1 - x_{PN}^1}{2L}\right),$$

$$\Omega_{TN}(x^1) \approx \omega\left(1 - \frac{x^1 - x_{TN}^1}{2L}\right),$$

where L is the characteristic scale on which the functions vary near resonant surfaces. In this case, the quasiclassical wave vector is described by the expression

$$k_1^2 = k_y^2\frac{x^1 - x_{PN}^1}{x_{TN}^1 - x^1}.$$

We can see that the transparency interval for the waves under study, $x_{PN}^1 < x^1 < x_{TN}^1$ (in which $k_1^2 > 0$), lies between two turning points, a usual point, $x^1 = x_{PN}^1$ (at which $k_1^2 = 0$) and a singular one, $x^1 = x_{TN}^1$ (at which $k_1^2 = \infty$). The region outside this interval between the magnetic shells is opaque to the waves. Hence, the interval $\Delta_N = x_{TN}^1 - x_{PN}^1$ determines the characteristic spatial scale on which the waves are localized across the magnetic shells. It is related to the polarization splitting of the spectrum by the relationship $\Delta_N = L\Delta\Omega_N/\Omega_{TN} \ll L$.

In the WKB approximation, the quantity $\Delta\Omega_N$ is described by the expression

$$\Delta\Omega_N = \frac{1}{2\pi N}\int_{l_-}^{l_+} A\{(\ln p)'' + p\beta_P - p^{-1}\beta_T\}dl'. \quad (15)$$

From Eqs. (4) and (6) for the toroidal and poloidal Alfvén waves, we can obtain an exact expression for the polarization splitting of the spectrum. We multiply Eq. (4) with $\bar{\omega} = \Omega_{PN}$ by $p^{-1}\Phi_{PN}$ and Eq. (6) with $\bar{\omega} = \Omega_{TN}$ by $p\Phi_{TN}$, take the difference of the two equations obtained, and integrate the resulting equation along a magnetic field line between the ionospheres of the northern and southern hemispheres. As a result, making use of the Hermitian nature of the operators $\hat{L}_T$ and $\hat{L}_P$, we obtain

$$\Omega_{TN}^2 - \Omega_{PN}^2 \quad (16)$$
$$= \int_{l_-}^{l_+}\Phi_{TN}\Phi_{PN}\{(\ln p)'' + p\beta_P - p^{-1}\beta_T\}dl \Big/ \int_{l_-}^{l_+}\frac{\Phi_{TN}\Phi_{PN}}{A^2}dl.$$

Expression (16) describes the polarization splitting of the spectrum of any mode of the standing Alfvén waves.

Expressions (15) and (16) differ from analogous expressions in the models with plasma at rest in that the value of $\Delta\Omega_N$ depends explicitly on the azimuthal wavenumber k_2. This implies that, when rotation of the magnetosphere plasma is taken into account, each azimuthal mode of the poloidal Alfvén waves corresponds to “its own” resonant magnetic shell. Or, in other words, at each magnetic shell, poloidal oscillations with different azimuthal mode numbers have different eigenfrequencies $\Omega_{PN} \equiv \Omega_{PN}(x^1, k_2)$.

6. NUMERICAL RESULTS AND DISCUSSIONS

It is convenient to solve Eqs. (4) and (6) numerically by passing from the orthogonal coordinate system (x^1, x^2, x^3) to the coordinate system (a, φ, θ) shown in Fig. 1. The equatorial radius of the magnetic shell, a, plays the role of the x^1 coordinate, the azimuthal angle φ is an analogue of the x^2 coordinate, and the latitude θ corresponds to the longitudinal coordinate x^3. Of course, the coordinate system (a, φ, θ) is nonorthogonal. In this system, the strength of the dipole magnetic field is given by the expression

$$B(a, \theta) = B(a_0, 0)\left(\frac{a_0}{a}\right)^3\frac{\sqrt{1 + 3\sin^2\theta}}{\cos^6\theta}.$$

The magnetic field line equation has the form

$$r = a\cos^2\theta,$$

where r is the position vector of a point on the field line, and the element of length in the longitudinal direction is equal to

$$dl = a\cos\theta\sqrt{1 + 3\sin^2\theta}d\theta.$$

Two of the three components of the metric tensor, g_1 and g_2, are expressed as

$$g_1 = \frac{\cos^6\theta}{1 + 3\sin^2\theta}, \quad g_2 = a^2\cos^6\theta.$$

Unfortunately, the expression for the component g_3 is more difficult to derive than those for g_1 and g_2. We can, however, find the ratio of the components $g_3(a, \theta)$ at the intersections of a magnetic surface x^3 = const with two

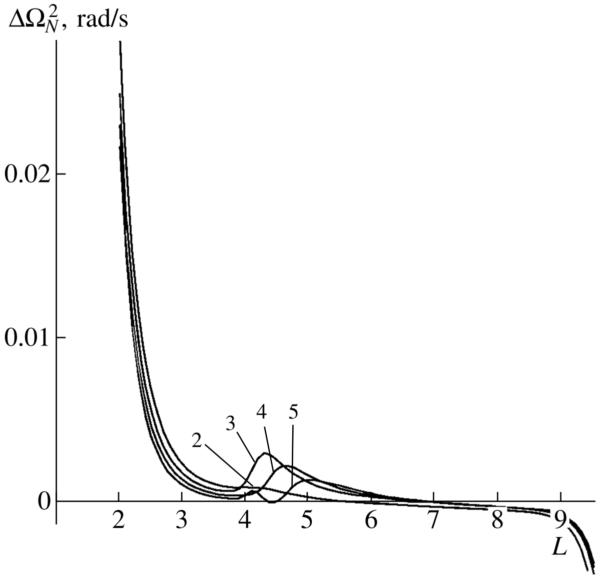

Fig. 5. Distribution of the polarization splitting of the spectrum, $\Delta\Omega_N^2$, for longitudinal modes with $N = 2$, 3, 4, and 5 and with the azimuthal mode number $m = -50$ across the magnetic shells.

neighboring magnetic shells a and a_0 at the latitudes θ and θ_0:

$$\frac{g_3(a, \theta)}{g_3(a_0, \theta_0)} = \left(\frac{a}{a_0}\right)^6 \left(\frac{\cos\theta}{\cos\theta_0}\right)^{12} \frac{1 + 3\sin^2\theta_0}{1 + 3\sin^2\theta}.$$

In the self-consistent magnetosphere model used here, the Alfvén velocity distribution $A(a, \theta)$ is determined by the condition for the rotating plasma to be in equilibrium. The equilibrium problem for such a plasma configuration was solved in [5], in which the following expression for the Alfvén velocity distribution in the geomagnetic meridian plane was obtained:

$$A(a, \theta) = A(\rho, 0)\frac{\Omega(a)}{\Omega(\rho)\sqrt{\beta(\theta)}},$$

where $\rho = a\cos^3\theta$ is the radial distance from a point (a, θ) on a magnetic field line to the symmetry axis, $\Omega(a)$ is the angular plasma velocity at a magnetic shell with the equatorial radius a, and

$$\beta(\theta) = \frac{(\cos\theta)^6}{1 + 3\sin^2\theta}.$$

The function $A(\rho, 0)$, which describes the Alfvén velocity distribution in the equatorial plane of the magnetosphere, can be specified based on the data from numerous satellite observations.

Figure 2 shows the longitudinal (along a magnetic field line) structure of the first three modes of the poloidal and toroidal standing Alfvén waves at the magnetic shell $L = 3$ (here, $L = a/R_E$, with R_E being the Earth's radius, is the dimensionless parameter labeling the magnetic shells). The oscillation amplitudes are normalized to unity. It should be noted that identical modes (i.e., those with the same number N) of the poloidal and toroidal standing Alfvén waves have very similar structures.

Figure 3 shows the eigenfrequencies of the fundamental mode of the poloidal and toroidal standing Alfvén waves. Curve *1* corresponds to toroidal waves with the azimuthal mode numbers $m = \pm 20$, 50, and 100, while curves *2–9* correspond to poloidal waves with the same azimuthal mode numbers. Strictly speaking, curves *2* and *3* are merely of methodological interest. Curve *2* illustrates how the poloidal eigenfrequency would behave in an immobile cold magnetosphere with the same Alfvén velocity distribution as that in the model under consideration. In this case, the inequality $\Omega_{T1} > \Omega_{P1}$ is satisfied for the entire magnetosphere. Curve *3* illustrates how the poloidal eigenfrequency $\Omega_{P1}(L)$ would behave in an immobile magnetosphere with a plasma having a finite pressure and obeying the same distribution as that in our model. The main difference from the cold plasma model is as follows: the plots of the functions $\Omega_{T1}(L)$ and $\Omega_{P1}(L)$ intersect one another in the outer magnetosphere; consequently, the inequality $\Omega_{T1} < \Omega_{P1}$ is satisfied in a region adjacent to the magnetosphere boundary. In these two cases, the poloidal eigenfrequency Ω_{P1} is independent of the azimuthal mode number m. Note that, for modes with arbitrary numbers N, the plot of the function Ω_{T1} is essentially the same as that in the immobile cold plasma model.

The eigenfrequencies $\Omega_{P1}(L)$ of the azimuthal modes with the numbers $m = -20$, 50, and 100 are shown graphically by curves *4–6* in Fig. 3. We can see that, in the transition region between the magnetospheric and solar wind plasmas, where the gradient of the angular velocity Ω is steepest, the eigenfrequencies of poloidal modes with different numbers m are split to a great extent. For $m < 0$, the functions $\Omega_{P1}(L)$ have local maxima in this region. It is known [14] that, in such regions, a resonant cavity can form for the poloidal Alfvén waves. Curves *7–9* correspond to the fundamental mode of poloidal standing Alfvén waves with the azimuthal mode numbers $m = 20$, 50 and 100. The main feature of these waves is that they are cut off at a certain magnetic shell $L = L_c$, whose position depends on the m value, and do not reach the magnetosphere boundary. The reason for this is that, for $L > L_c$, Eq. (6) has no solutions satisfying prescribed boundary conditions (9). This, in turn, is attributed to the fact that, for $m > 0$, the potential (the coefficient by the term free of derivatives) in Eq. (6) has a local minimum in the parameter $\bar{\varpi}$. As a result, there is no eigenvalue $\bar{\varpi}$ that satisfies the boundary conditions.

The upper curve in Fig. 3 shows the polarization splitting of the spectrum, $\Delta\Omega_1 = \Omega_{T1} - \Omega_{P1}$, for the fun-

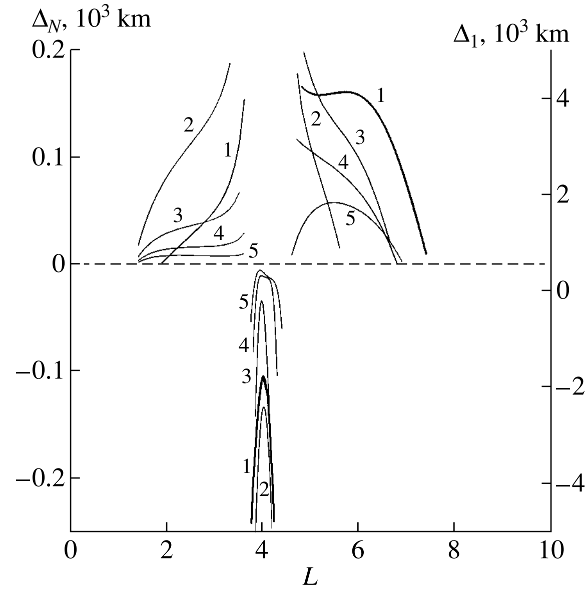

Fig. 6. Characteristic spatial scale Δ_N on which the first five ($N = 1, 2, 3, 4, 5$) standing Alfvén modes with the azimuthal mode number $m = 50$ are localized across the magnetic shells in the equatorial plane. The right ordinate corresponds to the $N = 1$ fundamental mode, while the left ordinate refers to the modes with $N = 2, 3, 4$, and 5.

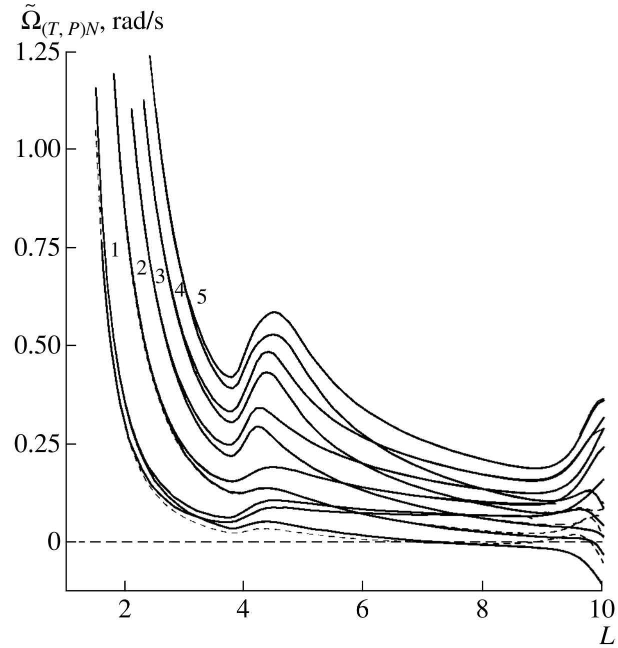

Fig. 7. Distribution of the toroidal and poloidal eigenfrequencies $\tilde{\Omega}_{(T, P)N} = \Omega_{(T, P)N} + m\Omega$ of the first five modes ($N = 1, 2, 3, 4, 5$) across the magnetic shells in a frame of reference in which the source is at rest.

damental mode of standing Alfvén waves with $m = -50$. The function $\Delta\Omega_1(L)$ changes its sign at the point of intersection of the functions $\Omega_{T1}(L)$ and $\Omega_{P1}(L)$. It should also be noted that the absolute value of $\Delta\Omega_1$ is one to two orders of magnitude larger than that for other modes with $N = 2, 3, \ldots$ (see Figs. 4, 5).

Figure 4 presents the eigenfrequencies $\Omega_{TN}(L)$ and $\Omega_{PN}(L)$ with $N = 1, 2, 3, 4$, and 5 for standing Alfvén waves with the azimuthal mode numbers $m = \pm 50$. Figure 5 displays the corresponding polarization splittings of the spectrum, $\Delta\Omega_N(L)$ with $N = 2, 3, 4$, and 5 for modes with $m = -50$. From Fig. 4 we can see that the value of $\Delta\Omega_1$ for the $N = 1$ fundamental mode is uniquely large. Figure 6 shows the corresponding functions Δ_N, which determine the characteristic equatorial dimensions of the regions where the waves under consideration are localized across the magnetic shells. The change in the signs of the functions Δ_N in the range $3.5 \lesssim L \lesssim 4.5$ is related to the change in the relative positions of the toroidal and poloidal surfaces. We see that the localization region of the fundamental mode of the standing Alfvén waves is one order of magnitude greater than those of their other modes.

The plots of the functions $\tilde{\Omega}_{PN} = \Omega_{PN} + m\Omega$ and $\tilde{\Omega}_{TN} = \Omega_{TN} + m\Omega$ in Fig. 7 show what the wave frequencies in a frame of reference in which the source is at rest should be in order for the Alfvén waves excited in the magnetosphere to have the frequencies Ω_{PN} and Ω_{TN}. Remember that the role of the source can be played by external currents in the ionosphere. The plots shown in Fig. 7 were calculated for the same azimuthal mode numbers, $m = \pm 50$, as those in Fig. 4. It is obvious that, the higher the plasma rotation velocity in the outer magnetosphere, the larger the difference in the frequencies of the source that excites eigenmodes with different azimuthal mode numbers m. In turn, the higher the number m, the greater this difference.

7. CONCLUSIONS

The main results of our study can be summarized as follows.

(i) We have derived equations (namely, Eqs. (4) and (6)) that describe the longitudinal (along the magnetic field lines) structure and spectrum of toroidally and poloidally polarized Alfvén waves in a dipole magnetosphere with a rotating plasma. We have solved these equations both analytically (in the WKB approximation) and numerically. The analytic solutions in the WKB approximation make it possible to describe the structure and spectrum of standing Alfvén waves with the mode numbers $N \gg 1$. For the first several modes ($N \sim 1$), Eqs. (4) and (6) were solved numerically.

(ii) We have constructed the spectra of the first five modes of the poloidal and toroidal standing Alfvén waves with different azimuthal mode numbers m. We have shown that plasma rotation results in an additional splitting of the spectrum of Alfvén eigenmodes. On each magnetic shell, not only do the toroidal and the poloidal waves have different eigenfrequencies but also the frequencies of poloidal waves with different azimuthal mode numbers m are different.

(iii) We have examined how the polarization splittings of the spectrum, $\Delta\Omega_N = \Omega_{TN} - \Omega_{PN}$ are related to the spatial scales on which the first several modes of the standing Alfvén waves are localized across the magnetic shells, Δ_N. We have shown that the value of $\Delta\Omega_1$ for the $N = 1$ fundamental mode is uniquely large: it is one to two orders of magnitude greater than the values of $\Delta\Omega_N$ for the next modes (those with $N = 2, 3, \ldots$). The polarization splitting is greatest in regions where the velocity gradient of the plasma rotation is maximum. In the Earth's magnetosphere, such regions are in the vicinities of the plasmapause and the magnetopause.

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