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THE FREQUENCY OF NONLINEAR PLASMA WAVES

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We study steady-state plane nonlinear plasma waves in a cold collisionless plasma in the absence of a magnetic field. The main conclusion following from the performed analysis is that allowance for the motion of the ion component of the plasma is of fundamental importance when studying nonlinear plasma waves. It is shown that, in general, the frequency of waves is essentially determined by the ion-component mass and depends equally on the velocity and amplitude of the waves.

1. INTRODUCTION

The properties of plasma waves of an infinitesimally small amplitude, i.e., linear waves, are studied fairly well [1, 2]. Recently, large-amplitude plasma waves have been studied rather extensively. Such waves are formed in a dense plasma as a result of the action of ultrarelativistic particle beams or the high-power laser radiation on the plasma, as well as during the transformation of intense electromagnetic waves incident on an inhomogeneous plasma into the plasma wave.

For the first time, the fundamental results for nonlinear waves in plasmas were obtained in [3–5] in which plane waves in an unbounded plasma were considered assuming that the plasma consists of cold electrons and infinitely heavy, motionless ions. Later, similar results for Langmuir waves were obtained independently in [6]. In [4–6], formulas for nonlinear Langmuir waves are obtained from which it follows that the wave frequency is a function of the limiting electron velocity in the wave. In this case, this velocity is an unknown constant. The dependence of the frequency of waves on their amplitude is not presented in explicit form in [4–6]. In this paper, we solve the problem in the same formulation as in [4–6] and obtain an exact and fairly simple formula for the frequency (see Eq. (13)), which rather clearly shows the dependence of the frequency of nonlinear Langmuir waves on their amplitude and velocity.

Developing the theory of A. I. Akhiezer et al. [4, 5], we took into account the motion of ions in the wave and obtained essentially new results which were given in [7, 8]. These results are of great importance in connection with the fact that for rather large amplitudes of the electric field in relativistic waves, it is necessary to allow for the motion of the ion component of the plasma, which has recently become clear. Such a condition follows from the works devoted to relativistic waves in plasmas [7–12] and from studies related to the interaction between the laser radiation and plasmas [13, 14].

In this paper, the dependence of the frequency of steady-state nonlinear Langmuir waves on the wave and plasma parameters is studied in detail with allowance for the motion of both the electron and ion components. As is noted in [3], development of the theory of nonlinear waves in plasmas meets with significant mathematical difficulties. In this paper, overcoming these difficulties by using quite acceptable simplifying assumptions, we were able to obtain analytical expressions for the frequency of nonlinear Langmuir waves. The expressions are rather simple and describe the behavior of the wave frequency in the entire range of

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the problem parameters. These expressions are obtained for both a typical plasma in which the ion mass is much greater than the electron mass (Eqs. (19), (21), and (22)) and an electron–positron plasma in which the masses of negatively and positively charged particles are identical (Eqs. (26) and (27)). All these formulas have been obtained for the first time.

The paper is organized as follows. In Sec. 2, we present the formulation of the problem and derive the basic equations required for solving the formulated problem. In Sec. 3, formulas for the wave frequency are obtained in various limiting cases. Section 4 presents the main conclusions.

2. FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

Let us consider qualitatively the process of the formation of a nonlinear plasma wave, following the reasoning of [1]. For simplicity, we consider a plasma with cold ions of one species and nonzero electron temperature T_e in the absence of an external magnetic field. Let a high-frequency plasma wave of an infinitesimally small amplitude begin to propagate in such a plasma. As is known [1, 2], for each plasma-wave harmonic with frequency ω and wave number k, the dispersion relation has the form

$$\omega(k) = \omega_{\mathrm{p}0}\,(1 + 3k^2 d_\mathrm{e}^2)^{1/2}, \tag{1}$$

where $\omega_{\mathrm{p}0} = (4\pi e^2 n_0/m)^{1/2}$ is the electron plasma frequency, $d_\mathrm{e} = [T_e/(4\pi e^2 n_0)]^{1/2}$ is the electron Debye length, n_0 is the unperturbed plasma density, and e and m are the elementary charge and the electron mass, respectively. It is usually assumed that $k^2 d_\mathrm{e}^2 \ll 1$, i.e., $\omega \approx \omega_{\mathrm{p}0}$. In this case, the phase velocity $v_\mathrm{ph} \approx \omega_{\mathrm{p}0}/k$ of the wave is much greater than the electron thermal velocity $v_{T\mathrm{e}} = (T_\mathrm{e}/m)^{1/2}$. The fulfillment of this condition is mandatory. Otherwise, the harmonic amplitude rapidly tends to zero due to collisionless Landau damping [1, 2].

With increasing the wave amplitude, one should take into account that due to the nonlinearity, the leading front of a traveling wave steepens, i.e., the higher harmonics appear. However, for plasma waves described by dispersion relation (1), the dispersion can stop the nonlinear steepening of the front and after some time, when the dispersion and nonlinear-steepening processes compensate for each other, the large-amplitude wave in the plasma can transform into a steady-state nonlinear wave [1].

In this paper, we consider steady-state periodic waves which are characterized by the following parameters: wavelength $\lambda = 2\pi/k$, wave oscillation period $T = 2\pi/\omega$, and wave phase velocity $u = \lambda/T = \omega/k$, where ω is the wave frequency. The problem consists in determining the dependence of the wave frequency ω on the plasma and wave parameters. Note that all the above-mentioned characteristics of waves correspond to the reference frame in which the unperturbed plasma rests. We call this frame the laboratory reference frame (LRF). In what follows we will perform our consideration in the reference frame coupled to the wave since the physical processes occurring in the steady-state wave are clearer in such a frame. In the wave reference frame, the plasma has the density $n = n_0\gamma$ and moves as a whole with respect to the motionless wave profile with velocity u. In this case, the spatial period of the wave is $\lambda_\mathrm{w} = \gamma\lambda$. Here, $\gamma = (1-\beta^2)^{-1/2}$, $\beta = u/c$, and c is the speed of light in free space.

If the condition $v_\mathrm{ph} \gg v_{T_\mathrm{e}}$ holds, then the plasma can be considered cold, which is assumed throughout the paper. Considering the one-dimensional case, we assume that the wave propagates in the direction opposite to the x-axis direction. In the wave reference frame in which the analyzed problem is stationary, all the desired variables describing the wave profile in the considered case are functions only of coordinate x. We will seek the solution in the form of a periodic alternating potential wave. In this case, the electric field has extreme values at the points between the maximum and the minimum of the potential on the scale equal to the wavelength λ_w. From the Maxwell equation

$$\frac{\mathrm{d}E(x)}{\mathrm{d}x} = 4\pi e\,[Zn_\mathrm{i}(x) - n_\mathrm{e}(x)] \tag{2}$$

for the electric field $E(x)$, it then follows that the right-hand side of Eq. (2) at these points is zero. Here,

$n_\mathrm{i}(x)$ and $n_\mathrm{e}(x)$ are the ion and electron number densities, respectively, and Z is the charge number of ions. Let the coordinate of one of the extreme points be $x = 0$ and let $Zn_\mathrm{i}(0) = n_\mathrm{e}(0) = n$ at this point. Let the extreme value of the electric field be $E(0) = E_0$. From Eq. (2), the relativistic equations of motion, and the continuity equations for electrons and ions, we have the total-momentum conservation law [8]

$$E^2(x)/(8\pi) - nu\,[p_\mathrm{e}(x) + p_\mathrm{i}(x)/Z] = E_0^2/(8\pi) - n\gamma\,(AM/Z + m)u^2. \quad (3)$$

Here, M is the rest mass of a proton, A is the atomic number of an ion, $p_\mathrm{e}(x) = mv_\mathrm{e}(x)\gamma_\mathrm{e}(x)$ and $p_\mathrm{i}(x) = AMv_\mathrm{i}(x)\gamma_\mathrm{i}(x)$ are the momenta of electrons and ions, respectively, v_i and v_e are the velocities of ions and electrons, respectively, $\gamma_\mathrm{i} = (1-(v_\mathrm{i}/c)^2)^{-1/2}$, $\gamma_\mathrm{e} = (1-(v_\mathrm{e}/c)^2)^{-1/2}$, and the constant is determined for $x = 0$. Equation (3) was obtained under the condition that a perturbed magnetic field is absent [8]: $n_\mathrm{i}(x)v_\mathrm{i}(x) = n_e(x)v_\mathrm{e}(x) = nu$. Note that with the obtained parameter γ, the considered problem has the physical meaning only for the velocity u not exceeding the speed of light in free space.

Using Eq. (3), we now obtain an expression for the frequency of oscillations in a wave. Denoting the electron plasma frequency in the wave reference frame by $\omega_\mathrm{pw} = (4\pi e^2 n/m)^{1/2}$, we introduce the dimensionless variables $\xi = x\omega_\mathrm{pw}\sqrt{\beta}/c$ and $\psi(\xi) = e\varphi(x)/(mc^2)$ for the coordinate and the potential, respectively. Then Eq. (3) in dimensionless variables can be represented by the formula

$$V(\psi,\gamma,\mu) = \varepsilon - (\mathrm{d}\psi(\xi)/\mathrm{d}\xi)^2/2 = \beta\mu\gamma - \sqrt{(\mu\gamma-\psi)^2-\mu^2} + \beta\gamma - \sqrt{(\gamma+\psi)^2-1}. \quad (4)$$

In what follows we will need another representation of the quantity $V(\psi,\gamma,\mu)$ in the form

$$V(\psi,\gamma,\mu) = \beta\mu\gamma\left(1-\sqrt{1-\frac{2\psi}{\beta^2\gamma\mu}+\frac{\psi^2}{\beta^2\gamma^2\mu^2}}\right) + \beta\gamma\left(1-\sqrt{1+\frac{2\psi}{\beta^2\gamma}+\frac{\psi^2}{\beta^2\gamma^2}}\right). \quad (5)$$

In Eqs. (4) and (5), the variable ψ is a function of ξ, i.e., $\psi = \psi(\xi)$, whereas other quantities are dimensionless and are as follows: $\beta = u/c$ is the wave phase velocity normalized to the speed of light in free space, $\gamma = (1-\beta^2)^{-1/2}$, $\mu = AM/(Zm)$, and $\varepsilon = (\mathrm{d}\psi/\mathrm{d}\xi)_0^2/2 = E_0^2/(8\pi nmcu)$ is the dimensionless energy density of the electric field at the point $\xi = 0$ at which $\psi = 0$ and the electric field is maximum. For convenience, we here give formulas for other parameters used in what follows. As we will see below, the product $\gamma\varepsilon$ is characteristic of the considered problem. Therefore, the special notation $\rho = \gamma\varepsilon$ is introduced for it. Another parameter which will be used is $\delta = \varepsilon/\varepsilon_\mathrm{m} = (E_0/E_{0\mathrm{m}})^2$, i.e., the ratio of the squared amplitude E_0^2 of the electric field to the squared limiting possible amplitude $E_{0\mathrm{m}}^2$ in the wave (a discussion of $E_{0\mathrm{m}}$ is presented below).

We now discuss the values which can be taken by the parameter $\mu = (A/Z)\,(M/m)$. It is easily seen that the parameter μ depends mainly on the species of ions of the plasma. In most typical cases, $\mu \gg 1$. For example, in an electron–proton plasma in which $A/Z = 1$, the parameter $\mu = M/m = 1838$. For a plasma consisting of ions that are heavier than protons, the ratio $A/Z \geq 2$ and the value of μ is greater. The exception is an electron–positron plasma in which $\mu = 1$. Allowing for this, we will further assume that $\mu \gg 1$ and introduce a small quantity $\theta = 1/\mu$ ($\theta \ll 1$). The special case $\mu = 1$ will be considered separately.

To find the wave frequency in the LRF, we use the formula $\omega = 2\pi u\gamma/\lambda_\mathrm{w}$. Here, λ_w, the spatial period of oscillations of the potential in the wave reference frame, is found from Eq. (4):

$$\lambda_\mathrm{w} = \frac{c}{\omega_\mathrm{pw}}\sqrt{\frac{2}{\beta}}\int\limits_{\psi_-}^{\psi_+}\frac{\mathrm{d}\psi}{\sqrt{\varepsilon - V(\psi,\gamma,\mu)}}\,,$$

where ψ_- and ψ_+ are roots of the equation $\varepsilon - V(\psi,\gamma,\mu) = 0$ and $V(\psi,\gamma,\mu)$ is determined by either Eq. (4)

or Eq. (5). Hence, for the quantity ω we obtain the relation

$$\omega = \omega(\varepsilon, \gamma, \mu) = \omega_{\rm p0}\pi\,\sqrt{2}\,(\beta\gamma)^{3/2}/J(\varepsilon, \gamma, \mu), \tag{6}$$

where $\omega_{\rm p0} = (4\pi e^2 n_0/m)^{1/2}$, n_0 is the plasma density in the LRF, and

$$J(\varepsilon, \gamma, \mu) = \int\limits_{\psi_-}^{\psi_+} \frac{{\rm d}\psi}{\sqrt{\varepsilon - V(\psi, \gamma, \mu)}}\,. \tag{7}$$

Equations (6) and (7) determine the desired frequency of oscillations of a longitudinal plasma wave in the most general form. It is seen that, first, ω depends on the wave characteristics, namely, the phase velocity u (parameter γ) and the electric-field amplitude E_0 (parameter ε), and, second, on the plasma characteristics, namely, the mass and charge of plasma particles (parameter μ) and the density n_0. The dependence of the frequency on the density is obvious. Therefore, we will not be interested in it and consider the dependence $\omega = \omega(\varepsilon, \gamma, \mu)$ as shown in Eq. (6).

First of all, we should introduce the nomenclature which will be used in the paper and is related to the parameters β and γ. By the nonrelativistic approximation, we mean the case where $\beta = 0$ and $\gamma = 1$. If the wave velocity is such that $\beta \ll 1$ and $\gamma < 2$, then we speak of the weakly relativistic case. The case where $\beta \approx 1$ and $\gamma \gg 1$ is called relativistic.

To find ω, one should evaluate integral (7). To do this, in turn, we should determine the integration limits ψ_- and ψ_+ and know in much detail the properties of the integrand determined mainly by the function $V(\psi, \gamma, \mu)$. By examining the properties of the function $V(\psi, \gamma, \mu)$, it can be shown [8] that the parameter ε has the limiting value

$$\varepsilon_{\rm m} = E_{0\rm m}^2/(8\pi nmcu) = \beta\gamma + \mu\beta\gamma - \sqrt{\mu^2\beta^2\gamma^2 + (\gamma - 1)\,(2\mu\gamma + \gamma - 1)}$$

above which nonlinear waves cannot exist for given values of n_0, γ, and μ. For $\mu \gg 1$, we then obtain

$$\varepsilon_{\rm m} \approx \left(1 + \frac{1}{2\mu\,(\gamma + 1)}\right)\frac{\beta\gamma}{\gamma + 1}\,.$$

It is seen from the formulas obtained for $\varepsilon_{\rm m}$ that the limiting amplitude of waves is mainly determined by the parameter γ, whereas one can neglect the dependence of $\varepsilon_{\rm m}$ on the parameter μ in the first approximation and put

$$\varepsilon_{\rm m} \approx \beta\gamma/(\gamma + 1) = (\gamma - 1)/\beta\gamma = [(\gamma - 1)/(\gamma + 1)]^{1/2}. \tag{8}$$

It follows from Eq. (8) that for weakly relativistic waves ($\beta \ll 1$), the parameter $\varepsilon_{\rm m} \approx \beta/2$ and, hence, $E_{0\rm m}^2 = 4\pi nmu^2$. For relativistic waves ($\gamma \gg 1$), we have $\varepsilon_{\rm m} \approx 1$ and $E_{0\rm m}^2 \approx 8\pi nmcu \approx 8\pi nmc^2 = 8\pi\gamma n_0 mc^2$. The existence of the limiting amplitude of nonlinear Langmuir waves seems to be related to the fact that for waves whose amplitude exceeds the limiting amplitude, the dispersion cannot stop the nonlinear steepening and the wave tilts.

For given parameters μ, γ, and ε, the variation range of oscillations of the potential is obtained from the equation

$$\beta\mu\gamma - \sqrt{(\mu\gamma - \psi)^2 - \mu^2} + \beta\gamma - \sqrt{(\gamma + \psi)^2 - 1} = \varepsilon,$$

whence one can find the desired quantities in general form, although the expressions for them turn out to be very cumbersome. Putting $\mu \gg 1$ and neglecting small terms, we obtain the approximate formulas

$$\psi_- \approx -\beta\mu\gamma^2\varepsilon/(\mu + 2\beta\gamma\varepsilon)\left(\sqrt{1 + 2\beta^2\,[1/(\beta\gamma\varepsilon) + 2/\mu]} - 1\right),$$

$$\psi_+ \approx \beta\mu\gamma^2\varepsilon/(\mu+2\beta\gamma\varepsilon)\left(\sqrt{1+2\beta^2\left[1/(\beta\gamma\varepsilon)+2/\mu\right]}+1\right)$$

for the "negative" and "positive" boundaries of the variation range of the potential in the wave. Allowing for Eq. (8), the product $\beta\gamma\varepsilon$ of parameters, which enters the formulas for ψ_- and ψ_+, can be represented as $\beta\gamma\varepsilon \approx \delta\,(\gamma-1)$. Since $\beta \le 1$ and $\varepsilon \le 1$, the product $\beta\gamma\varepsilon = \beta\rho$ can be much greater than unity only for $\gamma \gg 1$. In particular, the inequality $\beta\rho \gg \mu$ is possible only for $\gamma \gg \mu$. For weakly relativistic waves ($\beta \ll 1$), the condition $\beta\rho \approx \delta\,(\gamma-1) \ll 1$ always holds.

Let us determine the amplitudes ψ_- and ψ_+ for various relations between the parameters μ, ε, and γ. We start from the special case.

1. Let $\rho \approx \beta$. For weakly relativistic waves ($\beta \ll 1$), the parameter $\varepsilon \approx \beta \ll 1$. Hence, $\rho \ll 1$ and the amplitudes of the potential are written as

$$\psi_+ \approx (\beta^2/2)\,(2\sqrt{\delta}+\delta), \qquad \psi_- \approx -(\beta^2/2)\,(2\sqrt{\delta}-\delta). \tag{9}$$

For relativistic waves ($\gamma \gg 1$ and $\beta \approx 1$) in the considered case, we have $\rho \approx 1$. Hence, we obtain

$$\psi_+ \sim -\psi_- \approx \gamma \tag{10}$$

for any value of γ.

2. Let $\beta\rho \gg \mu$ ($\rho \gg \mu$). Under this condition, we obtain $\psi_- \approx -\gamma$ and $\psi_+ \approx \mu\gamma$. As expected, ψ_- and ψ_+ in this approximation are close to the limiting values $\psi_-^* = -(\gamma-1)$ and $\psi_+^* = \mu\,(\gamma-1)$, respectively [8]. It is easily seen that for $\beta\rho \approx \mu \gg 1$, the orders of the amplitudes ψ_- and ψ_+ are also comparable with ψ_-^* and ψ_+^*, respectively.

3. Let $\beta\rho \ll \mu$. Here, two cases are possible.

(a) $\rho \gg 1$. This means that $\gamma \gg 1$ ($\beta \approx 1$). In this case, the amplitudes are written as

$$\psi_+ \approx 2\beta\gamma\rho + \beta\gamma \approx 2\gamma\rho + \gamma, \qquad \psi_- \approx -\beta\gamma \approx -\gamma.$$

(b) $\rho \ll \beta \le 1$. In this case, we consider two possibilities.

I. $\gamma \gg 1$ and $\beta \approx 1$. In this case, $\varepsilon \ll 1/\gamma \ll 1$, so

$$\psi_+ \approx \gamma\sqrt{2\rho} + \gamma\rho \approx \gamma\sqrt{2\rho}, \qquad \psi_- \approx -\gamma\sqrt{2\rho} + \gamma\rho \approx -\gamma\sqrt{2\rho}. \tag{11}$$

II. $\gamma \approx 1$ and $\beta \ll 1$. In this case, $\varepsilon \le \varepsilon_{\mathrm{m}} \approx \beta/2 \ll 1$. This is the weakly relativistic case in which the formulas of Eq. (9) are valid. We emphasize that the parameter $\varepsilon \ll 1$ for any γ in the case $\rho \ll 1$.

3. DETERMINATION OF THE DEPENDENCE $\omega = \omega(\varepsilon, \gamma, \mu)$

We now proceed to finding analytical expressions for the wave frequency in various limiting cases. As was noted, in contrast to [3–6] in which ions were assumed motionless, we take into account the dynamics of ions in the wave. According to Eqs. (6) and (7), the fact that we allow for the motion of ions in the wave is seen in the dependence of the wave frequency on the parameter μ. It is this dependence that will be of primary importance for us. At first, we consider the limiting case $\mu \to \infty$. Then we consider waves in the plasma in which the parameter μ is finite, but large, i.e., $\mu \gg 1$. This is the most typical case if we bear in mind the space plasma or the plasma created under laboratory conditions. Finally, we separately consider electron–positron plasmas in which $\mu = 1$.

3.1. Approximation of motionless ions ($\theta = 1/\mu = 0$)

In this approximation, from Eq. (5) at the limit $\mu \to \infty$ we obtain

$$V_\infty(\psi, \gamma) = \beta\gamma - \sqrt{(\gamma+\psi)^2 - 1} + \psi/\beta, \tag{12}$$

where $V_\infty(\psi,\gamma) \equiv V(\psi,\gamma,\mu=\infty)$. By using the Euler substitution $\sqrt{(\gamma+\psi)^2-1} = x^2-(\gamma+\psi)$, integral (7) comprising the function $V_\infty(\psi,\gamma)$ takes the form

$$J_\infty = \int\limits_{\psi_-}^{\psi_+} \frac{\mathrm{d}\psi}{\sqrt{\varepsilon - V_\infty(\psi,\gamma)}} = \sqrt{\frac{2\beta}{1-\beta}} \int\limits_{b}^{a} \frac{(x^2-x^{-2})\,\mathrm{d}x}{\sqrt{(a^2-x^2)\,(x^2-b^2)}}\,,$$

where $a^2 = \gamma\,(1+\beta)\,(1+\beta\rho+\sqrt{\beta^2\rho^2+2\beta\rho})$ and $b^2 = \gamma\,(1+\beta)\,(1+\beta\rho-\sqrt{\beta^2\rho^2+2\beta\rho})$. The quantity J_∞ is expressed in terms of a complete elliptic integral of the second kind $\mathrm{E}(k)$:

$$J_\infty = (2\beta\gamma)^{3/2}\sqrt{\gamma(1-\beta)}\,a\mathrm{E}(k),$$

where $k = [1-(1+\beta\rho-\sqrt{\beta^2\rho^2+2\beta\rho})^2]^{1/2}$. Thus, the frequency in the approximation of motionless ions is represented by the formula

$$\omega(\varepsilon,\gamma) = \frac{\pi}{2}\omega_{\mathrm{p0}} \frac{\left(1+\beta\rho-\sqrt{\beta^2\rho^2+2\beta\rho}\right)^{1/2}}{\mathrm{E}(k)}, \tag{13}$$

where the product $\beta\rho$ takes the values from 0 to ∞.

We now proceed to analysis of Eq. (13) in detail. At first sight, the structure of Eq. (13) is complicated by the presence of the elliptic integral $\mathrm{E}(k)$. However, a closer examination shows that its influence is not significant. Indeed, if the quantity $\beta\rho$ varies from 0 to ∞, i.e., the modulus k varies from 0 to 1, then the value of the elliptic integral $\mathrm{E}(k)$ is between $\pi/2$ and 1, so that we may put $\mathrm{E}(k) \sim 1$ in the first approximation and write Eq. (13) in the form

$$\omega(\varepsilon,\gamma) \approx \frac{\pi}{2}\omega_{\mathrm{p0}} \left(1+\beta\rho-\sqrt{\beta^2\rho^2+2\beta\rho}\right)^{1/2}. \tag{14}$$

In the worst case, the frequency calculated from rather simple formula (14) differs from exact value (13) by the coefficient $\pi/2 \approx 1.6$ (i.e., by approximately 60%).

It follows from Eqs. (13) and (14) that the frequency decreases with increasing velocity and amplitude of waves. For waves propagating with the velocities $\beta \ll 1$, the quantity $\beta\rho \ll 1$. Putting $\gamma \approx 1$, from Eq. (13) we obtain

$$\omega(\varepsilon,\beta) \approx \omega_{\mathrm{p0}}\left(1-\frac{3}{8}\beta\varepsilon\right) \approx \omega_{\mathrm{p0}}\left(1-\frac{3}{16}\beta^2\delta\right). \tag{15}$$

It is seen that in this case, the frequency differs only slightly from ω_{p0}. With increasing the quantity $\beta\rho$ from 0 to 1, i.e., for $\beta\rho \le 1$, the frequency ω remains close to ω_{p0} (e.g., for $\beta\rho = 1$, the wave frequency $\omega \approx 0.7\omega_{\mathrm{p0}}$). Since $\beta\rho \approx \delta\,(\gamma-1)$, the condition $\beta\rho \le 1$ can be written as $\gamma - 1 \le 1/\delta$. It follows from the latter relation that for waves with the limiting possible amplitude, i.e., for $\delta = 1$, the parameter $\gamma \le 2$. But if $\delta \ll 1$, then the case where $\gamma \gg 1$ is possible. Hence, we arrive at an interesting conclusion, namely, the frequency of plasma waves is close to the frequency ω_{p0} of linear oscillations in the plasma not only for waves having a small velocity ($\beta \ll 1$), but also for waves traveling with subluminal velocities and having a small amplitude of the electric field compared with the limiting amplitude. Actually, this conclusion is a consequence of the fact that, according to Eq. (13), the frequency depends on $\beta\rho \approx \delta\,(\gamma-1)$, i.e., on the product of the parameter proportional to the wave amplitude and the parameter dependent on the wave velocity.

With the further increase in the parameter $\beta\rho$, when it becomes greater than unity, the elliptic integral $\mathrm{E}(k)$ differs from unity by a value smaller than 10% (e.g., for $\beta\rho = 1$, the quantity $\mathrm{E}(k) \approx 1.08$). Therefore, the wave frequency for $\beta\rho > 1$ can be calculated with good accuracy from Eq. (14). For relativistic waves

($\gamma \gg 1$) when $\beta\rho \approx \gamma\varepsilon \gg 1$, the frequency is determined by the expression

$$\omega(\varepsilon,\gamma) \approx \omega_{\mathrm{p0}}\pi/(2\sqrt{2\gamma\varepsilon}). \tag{16}$$

In this case, if the amplitude of waves is nonzero and their velocity approaches the speed of light in free space ($\beta \to 1$ and $\gamma \to \infty$), the frequency of waves tends to zero.

To conclude this section, we note that, as expected, Eq. (13) yields the results obtained for the first time by A. I. Akhiezer et al. in the approximation of infinitely heavy ions for two limiting cases, namely, (i) the nonrelativistic approximation within the framework of which $\omega = \omega_{\mathrm{p0}}$ [3] and (ii) the case of relativistic waves ($\gamma \gg 1$) with the limiting amplitudes ($\varepsilon \approx 1$), for which the frequency $\omega(\gamma) \approx \omega_{\mathrm{p0}}\pi/(2\sqrt{2\gamma})$ [4, 5]. It is seen that in the first case, the frequency does not depend on both the velocity and the wave amplitude and is determined by the frequency of linear oscillations in the plasma. In the second case, the frequency monotonically decreases with increasing wave velocity.

3.2. The most widespread case $\mu \gg 1$

Our main purpose in this section is to reveal the dependence of the frequency on the parameter μ, assuming that μ is large, but finite, i.e., $\mu \gg 1$. The fact that μ is large is exactly the reason from which it is intuitively clear that the above-considered approximation not allowing for the dynamics of ions should be applicable for certain values of ε and γ in this case. Indeed, it is easy to show that this approximation is suitable for studying waves propagating with such velocities for which the condition $\gamma \ll \mu$ or even the less stringent condition $\gamma < \mu$ holds for any ε. Indeed, if these inequalities hold, then the integration limits in Eq. (7) as functions of γ are determined by Eqs. (9)–(11). In this case, for the function $V(\psi,\gamma,\mu)$ determined by Eq. (5), the terms entering the radicand and comprising the parameter μ are much less than unity. Representing the square root as a series and neglecting small terms comprising the second and higher powers of ψ, we obtain that $V(\psi,\gamma,\mu) \approx V_\infty(\psi,\gamma)$, where $V_\infty(\psi,\gamma)$ is determined by Eq. (12), i.e., for the frequency we arrive at Eq. (13) which is valid in the approximation of infinitely heavy ions.

Taking into account these considerations, we first analyze the behavior of waves traveling with small velocities. Then we consider the properties of waves propagating with relativistic velocities. In this case, we reveal what contribution to the wave frequency is given by allowance for finite values of μ.

3.2.1. Weakly relativistic waves ($\beta \ll 1$)

In a rough approximation, the results of Sec. 3.1. can certainly be applied for waves with small velocities, i.e., if $\gamma \approx 1 \ll \mu$. However, we make an attempt to reveal the frequency variation if the finite value of μ is allowed for, as well as the trend of this variation. Taking into account that in this case, the limits of integration in Eq. (7) are determined by the formulas in Eq. (9) and, hence, the values of the variable ψ in the integrand of Eq. (7) are much less than unity, the function $V(\psi,\gamma,\mu)$ is represented as

$$V(\psi,\gamma,\theta) = \beta\gamma - \sqrt{(\gamma+\psi)^2-1} + \psi/\beta + \theta\psi^2/(2\beta^3\gamma^3), \tag{17}$$

where $\theta = 1/\mu \ll 1$. Then, considering the term with the parameter θ in Eq. (17) a small addition, we take a Taylor series expansion of integral (7) comprising function (17) and denoted by $J(\varepsilon,\gamma,\theta)$ in the vicinity of the point $\theta = 0$ and limit the series sum to the term proportional to θ:

$$J(\varepsilon,\gamma,\theta) = J(\varepsilon,\gamma,0) + \theta\left[\frac{\partial J(\varepsilon,\gamma,\theta)}{\partial\theta}\right]_{\theta=0}. \tag{18}$$

The first term $J(\varepsilon,\gamma,0) = J_\infty$ of series (18) was found in Sec. 3.1. The derivative

$$\frac{\partial J(\varepsilon,\gamma,\theta)}{\partial\theta} = \frac{\partial}{\partial\theta}\int\limits_{\psi_-}^{\psi_+}\frac{\mathrm{d}\psi}{\sqrt{\varepsilon - V(\psi,\gamma,\theta)}}$$

is represented as

$$\left[\frac{\partial}{\partial\theta}\int\limits_{\psi_-}^{\psi_+}\frac{\mathrm{d}\psi}{\sqrt{\varepsilon - V(\psi,\gamma,\theta)}}\right]_{\theta=0} = \left[2\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varepsilon}\int\limits_{\psi_-}^{\psi_+}\sqrt{\varepsilon - V(\psi,\gamma,\theta)}\,\mathrm{d}\psi\right]_{\theta=0}$$
$$= -\frac{\partial}{\partial\varepsilon}\left[\int\limits_{\psi_-}^{\psi_+}\frac{[\partial V(\psi,\gamma,\theta)/\partial\theta]\,\mathrm{d}\psi}{\sqrt{\varepsilon - V(\psi,\gamma,\theta)}}\right]_{\theta=0} = -\frac{1}{2\beta^3\gamma^3}\frac{\partial I}{\partial\varepsilon},$$

where

$$I = \int\limits_{\psi_-}^{\psi_+}\frac{\psi^2\,\mathrm{d}\psi}{\sqrt{\varepsilon - V_\infty(\psi,\gamma)}}.$$

Such a representation is possible due to the following circumstances. First, the variables θ and ε are independent. Second, differentiating the integral

$$\int\limits_{\psi_-}^{\psi_+}\sqrt{\varepsilon - V(\psi,\gamma,\theta)}\,\mathrm{d}\psi$$

with respect to the variables θ and ε is reduced to differentiating the integrand since this function is zero at the integration limits.

The integral I in which the function $V_\infty(\psi,\gamma)$ is determined by Eq. (12) can be evaluated similarly to J_∞. As a result, we obtain

$$I = \sqrt{2\beta\,(1+\beta)}[(Y_3 - Y_{-3})/4 - \gamma\,(Y_2 - Y_{-2}) + (\gamma^2 + 1/4)\,(Y_1 - Y_{-1})],$$

where

$$Y_n = \int\limits_b^a\frac{x^{2n}\,\mathrm{d}x}{\sqrt{(a^2 - x^2)\,(x^2 - b^2)}}.$$

Here, the quantities a and b defined in Sec. 3.1 are used. The integrals Y_n are expressed in terms of complete elliptic integrals of the first and second kinds $\mathrm{K}(k)$ and $\mathrm{E}(k)$, respectively, where the parameter k is the same as in Sec. 3.1. Rejecting small terms comprising the parameters β and ε to powers higher than three and one, respectively, we obtain the derivative

$$\left[\frac{\partial J(\varepsilon,\gamma,\theta)}{\partial\theta}\right]_{\theta=0} \approx \frac{\gamma^{3/2}\sqrt{2\beta}[\mathrm{E} - \mathrm{K} + \mathrm{E}\sqrt{2\beta\varepsilon} + (9\mathrm{E} - \mathrm{K})\,\beta\varepsilon + 2\,(4\mathrm{E} - \mathrm{K})\,\beta^2/3 + 8\beta^2\sqrt{2\beta\varepsilon}\,\mathrm{E}/3]}{2\beta\,(1 + \beta\varepsilon/2)\sqrt{1 + \beta\varepsilon + \sqrt{2\beta\varepsilon}}}.$$

Here, we omitted the argument k of the elliptic integrals $\mathrm{K}(k)$ and $\mathrm{E}(k)$. Then, taking into account that for $\beta \ll 1$, the relation $\varepsilon = \beta\delta/2$ holds and the modulus $k \ll 1$, we use the asymptotic expansion of the elliptic integrals entering the expression for the derivative for small k. Then, substituting the derivative into

Eq. (18) and allowing for Eq. (15), for the frequency we finally obtain

$$\omega(\beta, \delta, \theta) \approx \omega_{\mathrm{p}} \left(1 - \frac{3}{16}\beta^2 \delta + \frac{15}{16}\theta\delta\right), \tag{19}$$

where $\omega_{\mathrm{p}} = \omega_{\mathrm{p}0}\,(1 + \theta)^{1/2}$ is the frequency of linear oscillations of the plasma with allowance for the ion mass [2].

Thus, allowance for the motion of ions gives an increase in the frequency due to the positive small addition (the third term in the parentheses of Eq. (19)) which is proportional to the wave amplitude. Although the frequency of weakly relativistic waves almost does not differ from $\omega_{\mathrm{p}0}$, dependence (19) is interesting in the sense that it allows one to understand the influence of the nonlinearity and the dynamics of ions on the wave frequency. Indeed, it is seen that allowance for the nonlinearity (the second term in the parentheses of Eq. (19)) leads to a decrease in the frequency, whereas allowance for the dynamics of ions (the third term), on the contrary, leads to an increase in the frequency of weakly relativistic waves. It is interesting to note that for the wave velocity $\beta = \sqrt{5\theta}$, the influence of the nonlinearity on the frequency is compensated by the influence of the dynamics of ions, and the frequency of nonlinear waves is equal to the frequency ω_{p} of linear oscillations of the plasma.

If we put $\beta = \mathrm{const}$, then it follows from Eq. (19) that for the velocities $\beta < \sqrt{5\theta}$, the frequency is higher than ω_{p} and increases with increasing wave amplitude, whereas for $\beta > \sqrt{5\theta}$, the frequency is lower than ω_{p} and decreases with increasing amplitude δ. For a fixed wave amplitude ($\delta = \mathrm{const}$), we now consider the most interesting case (from our viewpoint) where the wave amplitude is equal to the limiting possible amplitude, i.e., $\delta = 1$. In this case, from Eq. (19) we obtain that for $\beta = 0$, the frequency is higher than ω_p and is equal to its maximum value $\omega \approx \omega_{\mathrm{p}}\,(1 + \theta) = \omega_{\mathrm{p}0}\,(1 + \theta)^{3/2}$. Then, with increasing velocity, the wave frequency decreases. For the velocity $\beta = \sqrt{5\theta}$, it becomes equal to $\omega_{\mathrm{p}} = \omega_{\mathrm{p}0}\,(1 + \theta)^{1/2}$ and then decreases. In conclusion, we note that for $\beta = 0$, the result presented in [15] follows from Eq. (19).

3.2.2. Relativistic waves for $\rho \gg 1$

Before proceeding to the relativistic case, we verify that the waves for which the condition $\beta\rho \le 1$ holds have frequencies close to $\omega_{\mathrm{p}0}$, which takes place in the approximation of infinitely heavy ions (Sec. 3.1). Indeed, in this case, the integration limits in integral (7) are determined by Eq. (10). It is easily seen that the terms entering the radicand of Eq. (5) and comprising the parameter μ are much less than unity on the integration interval. Therefore, the function $V(\psi, \gamma, \mu)$ can be replaced by $V_\infty(\psi, \gamma)$. Thus, the approximation of motionless ions is applicable here and all the conclusions presented in Sec. 3.1 for the case $\beta\rho \le 1$ remain valid.

For relativistic waves, we will thus assume that $\beta\rho = \beta\gamma\varepsilon \gg 1$, which is equivalent to the inequalities $\rho \gg 1$ and $\gamma \gg 1$ since $\beta \approx 1$ and $\varepsilon \le 1$. In this case, putting $\sqrt{(\gamma + \psi)^2 - 1} \approx \gamma + \psi$, the function $V(\psi, \gamma, \mu)$ in integral (7) is represented on the integration interval $0 \le \psi \le \psi_+$ as

$$V(\psi, \gamma, \mu) \approx \beta\mu\gamma - \gamma\,(1 - \beta) - \psi - \sqrt{(\mu\gamma - \psi)^2 - \mu^2}\,. \tag{20}$$

On the interval $0 \ge \psi \ge \psi_-$, allowing for the fact that $|\psi_-| \le \gamma - 1$, the function $V(\psi, \gamma, \mu)$ in all cases can be replaced with sufficient accuracy by the function $V_\infty(\psi, \gamma)$ determined by Eq. (12). Thus, integral (7) in this case can be represented as the sum $J(\varepsilon, \gamma, \mu) = J_1 + J_2$ of two integrals, where

$$J_1 = \int\limits_{\psi_-}^{0} \frac{\mathrm{d}\psi}{\sqrt{\varepsilon - V_\infty(\psi, \gamma)}}\,, \qquad J_2 = \int\limits_{0}^{\psi_+} \frac{\mathrm{d}\psi}{\sqrt{\varepsilon - V(\psi, \gamma, \mu)}}\,.$$

The function $V(\psi, \gamma, \mu)$ in the integral J_2 is determined by Eq. (20).

Evaluation of the integral J_1 is similar to that of J_∞. As a result, J_1 is expressed in terms of elliptic integrals of the second kind as follows:

$$J_1 = \sqrt{\frac{2\beta}{1-\beta}}\left\{a\left[\mathrm{E}(k) - \mathrm{E}(q,k)\right] - \mathrm{E}(p,k)/(ab^2)\right\}.$$

Here,

$$q = \arcsin\sqrt{\frac{a^2-s^2}{a^2-b^2}}\,, \qquad p = \arcsin\left(\frac{a}{s}\sqrt{\frac{s^2-b^2}{a^2-b^2}}\right), \qquad s^2 = \gamma\,(1+\beta),$$

and the parameters a, b, and k are defined in Sec. 3.1. The final evaluation of the integral J_1 yields $J_1 \approx \gamma/\sqrt{\varepsilon}$ (recall that in the considered case $\gamma \gg 1$, we have $\rho = \gamma\varepsilon \gg 1$, i.e., ε cannot take a zero value: $\varepsilon \gg 1/\gamma$).

We now evaluate the integral J_2. The function $V(\psi,\gamma,\mu)$ determined by Eq. (20) is represented as follows:

$$V(\psi,\gamma,\mu) \approx \mu\left[\beta\gamma - \psi/\mu - \sqrt{(\gamma-\psi/\mu)^2 - 1}\right].$$

Introduce the notation $y = \psi/\mu$. Using the change of variables

$$\gamma - t = y + \sqrt{(\gamma-y)^2 - 1},$$

the integral J_2 is expressed in terms of the table integrals as

$$J_2 = \frac{\sqrt{\mu}}{2}\left(\int\limits_g^h \frac{t^{-2}\,\mathrm{d}t}{\sqrt{h-t}} - \int\limits_g^h \frac{\mathrm{d}t}{\sqrt{h-t}}\right) = \frac{\sqrt{\mu}}{2}\left(\frac{\sqrt{h-g}}{(gh)} - 2\sqrt{h-g} - \frac{1}{2h^{3/2}}\ln\frac{\sqrt{h}-\sqrt{h-g}}{\sqrt{h}+\sqrt{h-g}}\right),$$

where

$$g = \gamma\,(1-\beta) = \frac{1}{\gamma\,(1+\beta)} \approx \frac{1}{2\gamma}, \qquad h = g + \varepsilon/\mu.$$

Thus, evaluation of the integral J yields

$$J(\varepsilon,\gamma,\mu) = \frac{2\mu\gamma^{3/2}\sqrt{\rho}}{\mu+2\rho}\left(1 + \frac{\mu+2\rho}{2\mu\rho} - \frac{\mu+2\rho}{2\mu\gamma^2} + \frac{\mu}{2\sqrt{2\rho\,(\mu+2\rho)}}\ln\frac{\sqrt{1+\mu/(2\rho)}+1}{\sqrt{1+\mu/(2\rho)}-1}\right).$$

It is easily seen that in the considered approximation ($\rho \gg 1$ and $\mu \gg 1$), the second and third terms in the parentheses on the right-hand side of the obtained expression are much less than unity. Omitting these terms and substituting the obtained value of the integral $J(\varepsilon,\gamma,\mu)$ into Eq. (6), we arrive at the formula

$$\omega(\varepsilon,\gamma,\mu) \approx \omega_{\mathrm{p}0}\pi\,(\mu+2\rho) \bigg/ \left[\mu\sqrt{2\rho}\left(1 + \frac{\mu}{2\sqrt{2\rho\,(\mu+2\rho)}}\ln\frac{\sqrt{1+\mu/(2\rho)}+1}{\sqrt{1+\mu/(2\rho)}-1}\right)\right] \tag{21}$$

for the frequency.

Let us consider the dependence of the frequency on the relation between the parameters ρ and μ. For $1 \ll \rho \ll \mu$, the second term in the parentheses in the denominator of Eq. (21) is equal to unity and Eq. (21) is transformed into Eq. (16), as expected. In the case $\rho \geq \mu \gg 1$, the second term can be omitted since it is small in comparison with unity, and we obtain the formula

$$\omega(\varepsilon,\gamma,\mu) \approx \omega_{\mathrm{p}0}\pi\frac{\mu+2\rho}{\mu\sqrt{2\rho}} = \omega_{\mathrm{p}0}\pi\frac{\mu+2\gamma\varepsilon}{\mu\sqrt{2\gamma\varepsilon}} \tag{22}$$

for the frequency. From Eq. (22), we have an important result consisting in that in this case, the frequency depends on all three parameters of the problem, i.e., ε, γ, and μ, and the dependence on all the parameters, including μ, is essential. Thus, we obtain that for $1 \ll \rho \ll \mu$, the wave frequency is expressed by Eq. (16), i.e.,

$$\omega(\varepsilon, \gamma) \approx \omega_{\mathrm{p0}}\pi/(2\sqrt{2\rho}) = \omega_{\mathrm{p0}}\pi/(2\sqrt{2\gamma\varepsilon}),$$

which is obtained in the approximation of infinitely heavy ions. In this case, the wave frequency is lower than ω_{p0}, does not depend on μ, and decreases with increasing ρ. For $\rho \gg \mu$, from Eq. (22) we obtain

$$\omega(\varepsilon, \gamma) \approx \omega_{\mathrm{p0}}\pi\sqrt{2\rho}/\mu = \omega_{\mathrm{p0}}\pi\sqrt{2\gamma\varepsilon}/\mu,$$

i.e., the frequency essentially depends on μ and, on the contrary, increases with increasing ρ. If the velocity of finite-amplitude waves tends to the speed of light in free space, which is equivalent to the limit $\rho \to \infty$, then the frequency tends to infinity, which is fully opposite to the frequency behavior obtained in the approximation of motionless ions, within the framework of which the frequency decreases to zero for $\beta \to 1$.

As a result, we obtain that for weakly relativistic and relativistic waves in the case $\rho = \gamma\varepsilon \le \beta \le 1$, the wave frequency is close to ω_{p0}. For $\rho \gg 1$ and any relation between μ and ρ, very simple formula (22) can be used for the wave frequency with accuracy up to 50%. This formula correctly represents the functional dependence of the frequency on the parameters ε, γ, and μ, i.e., gives a decrease in the frequency with increasing ρ for $\rho \ll \mu$ and describes an increase in the frequency with increasing ρ for $\rho \gg \mu$.

For a fixed value of μ, the dependence of ω on the parameter ρ, expressed by Eq. (22), means that the frequency has the minimum value ω_{min} at a certain value of ρ. Under the condition $\partial\omega/\partial\rho = 0$, it follows from Eq. (22) that $\omega_{\mathrm{min}} \approx 2\pi\omega_{\mathrm{p0}}/\sqrt{\mu}$ for $\rho_{\mathrm{min}} \approx \mu/2$. We emphasize that the ratio $\omega_{\mathrm{min}}/\omega_{\mathrm{p0}}$ depends only on μ. It is evident that at certain $\rho = \rho_0 \gg \mu$, the frequency ω is again equal to the plasma frequency ω_{p}, as for linear waves. The value of ρ_0 for which $\omega = \omega_{\mathrm{p}}$ is found by using Eq. (22): $\rho_0 \approx \mu^2/(2\pi^2)$. Thus, for a fixed amplitude of the electric field of the wave and a variation in the wave velocity from zero to the speed of light, the frequency initially decreases to a certain minimum value and then monotonically and indefinitely increases. In this case, the frequency twice takes the value $\omega = \omega_{\mathrm{p}}$ on the interval $0 \le u \le c$. At first, this occurs at the drop phase and then, during an increase from the minimum value to infinity. At the drop phase, for $\rho < \rho_{\mathrm{min}} \approx \mu/2$, one can use the formulas for the frequency which were obtained in the case of infinitely heavy, motionless ions (Eqs. (13)–(16)). If $\rho > \rho_{\mathrm{min}} \approx \mu/2$, then the frequency should be found from the relations obtained with allowance for the motion of ions (Eqs. (21) and (22)).

Let us estimate the amplitude of the electric field of a nonlinear wave, under which the wave begins to "respond" to the dynamics of ions. We take $\rho = \rho_{\mathrm{min}} \approx \mu/2$ and assume that the wave propagates in an electron–proton plasma and its amplitude is close to the limiting amplitude ($\varepsilon \approx \varepsilon_{\mathrm{m}}$). Then, taking into account that $\rho_{\mathrm{min}} \gg 1$, from Eq. (8) we have $\varepsilon_{\mathrm{m}} \approx 1$, whence the estimate $E_{0\mathrm{m}} \approx \sqrt{4\pi n_0 M c^2}$ for the electric-field amplitude follows. From this formula, we obtain $E_{0\mathrm{m}} \approx 10^{12}$ V/m for a plasma with the density $n_0 \sim 10^{18}$ cm^{-3}. Such an amplitude of the electric field corresponds to the value observed in the laser beam with a wavelength of 1 μm and an intensity of 10^{18} W/cm^2.

3.3. Electron–positron plasma ($\mu = 1$)

In this case, formula (6) for the frequency is represented as

$$\omega = \omega(\varepsilon, \gamma) = \frac{\pi}{2}\omega_{\mathrm{pe}}\,(\beta\gamma)^{3/2}/J(\varepsilon, \gamma), \tag{23}$$

where $\omega_{\mathrm{pe}} = (8\pi e^2 n_0/m)^{1/2}$ is the frequency of linear oscillations of the plasma and

$$J(\varepsilon, \gamma) = \int\limits_0^{\psi_+} \frac{\mathrm{d}\psi}{\sqrt{\varepsilon - V(\psi, \gamma)}}\,. \tag{24}$$

The upper limit in the integral $J(\varepsilon, \gamma)$ is given by the formula

$$\psi_+ = (\beta\gamma - \varepsilon/2)\big\{\varepsilon(\beta\gamma - \varepsilon/4)/[1 + \varepsilon\,(\beta\gamma - \varepsilon/4)]\big\}^{1/2},$$

whereas the function $V(\psi, \gamma)$ is given by the expression

$$V(\psi, \gamma) = 2\beta\gamma - \sqrt{(\gamma - \psi)^2 - 1} - \sqrt{(\gamma + \psi)^2 - 1}\,.$$

The effective potential well described by the function $V(\psi, \gamma)$ has a shape symmetric with respect to the point $\psi = 0$. The limiting depth of the well is $\varepsilon_{\rm m} = 2\beta\gamma\,(1 - \sqrt{\gamma/(\gamma+1)})$ and the limiting amplitudes of the potential are $\psi_-^* = -(\gamma - 1)$ and $\psi_+^* = \gamma - 1$ [7].

3.3.1. Weakly relativistic waves ($\beta \ll 1$)

For weakly relativistic waves, the depth of the potential well is $\varepsilon_{\rm m} \approx \beta\,(2 - \sqrt{2}) \ll 1$, i.e., $\varepsilon \le \varepsilon_{\rm m} \ll 1$. To find an expression for the frequency of waves in the case $\beta \ll 1$, we make the change of variables

$$t = \left[\sqrt{(\gamma - \psi)^2 - 1} + \sqrt{(\gamma + \psi)^2 - 1}\right]/2$$

in integral (24). Then integral (24) takes the form

$$J(\varepsilon, \gamma) = \frac{1}{\sqrt{2}}\left(\beta^2\gamma\int_b^{\beta\gamma}\frac{(1 - t^2/\gamma^2)^{-1/2}\,{\rm d}t}{\sqrt{(\beta^2\gamma^2 - t^2)\,(b - t)}} - \gamma\int_b^{\beta\gamma}\frac{(1 - t^2/\gamma^2)^{-3/2}\sqrt{\beta^2\gamma^2 - t^2}\,{\rm d}t}{\sqrt{b - t}}\right),$$

where $b = \beta\gamma - \varepsilon/2$. We represent the expressions $(1 - t^2/\gamma^2)^{-1/2}$ and $(1 - t^2/\gamma^2)^{-3/2}$ entering the integrands by power series. Since $t^2/\gamma^2 \ll 1$ on the integration interval, we can limit ourselves to the finite number of terms in the obtained series. As a result, we have

$$J(\varepsilon, \gamma) = \frac{1}{\sqrt{2}}\left(\int_b^{\beta\gamma}\frac{\beta^2\gamma^2\,{\rm d}t}{\sqrt{(\beta^2\gamma^2 - t^2)\,(b - t)}} - \sum_n a_n(\varepsilon, \gamma)\int_b^{\beta\gamma}\frac{(\beta^2\gamma^2 - t^2)^{n+1/2}\,{\rm d}t}{\sqrt{b - t}}\right), \tag{25}$$

where $n = 0, 1, 2, \ldots$ and $a_n(\varepsilon, \gamma)$ are the series coefficients dependent on ε and γ. We limit ourselves to the approximation within the framework of which we omit the terms entering the expression for the integral $J(\varepsilon, \gamma)$ and comprising the small parameter β to the power higher than four. An examination shows that in this approximation, it is sufficient to retain the first term of the sum in Eq. (25). In this approximation, we obtain

$$J(\varepsilon, \gamma) \approx (\beta\gamma)^{3/2}\left\{{\rm K}(k) - \left(1 + \frac{3}{4}\beta^2\right)\frac{8}{3}\left[\left(1 - \frac{\nu}{4}\right){\rm K}(k) - \left(1 - \frac{\nu}{2}\right){\rm E}(k)\right]\right\},$$

where ${\rm K}(k)$ and ${\rm E}(k)$ are complete elliptic integrals of the first and second kinds, respectively, $k = \sqrt{\nu}/2$, and $\nu = \varepsilon/\beta = \delta\,(2 - \sqrt{2}) \approx 3\delta/5$. Using the asymptotic expansions of the integrals ${\rm K}(k)$ and ${\rm E}(k)$ for small k, we finally obtain

$$J(\varepsilon, \gamma) \approx \frac{\pi}{2}(\beta\gamma)^{3/2}\left(1 - \frac{15}{16}\nu - \frac{3}{4}\beta^2\nu\right).$$

Substitution of the relation obtained for $J(\varepsilon, \gamma)$ into Eq. (23) yields

$$\omega(\delta) \approx \omega_{\rm pe}\left(1 + \nu + \frac{3}{4}\beta^2\nu\right) \approx \omega_{\rm pe}\left(1 + \frac{3}{5}\delta + \frac{1}{2}\delta\beta^2\right). \tag{26}$$

It is seen that Eq. (26) for the frequency is similar in form to Eq. (19) obtained for weakly relativistic waves in a plasma containing heavy ions, but has two important differences. The first difference is that the term comprising the parameter β, which appears in formula (26) for the frequency due to allowance for the nonlinearity, is positive. The second difference is that the addition to the frequency, which is related only to the wave amplitude (the second term in the parentheses of Eq. (26)), is important for waves whose amplitude is close to the limiting amplitude. Due to this fact, for, e.g., $\delta = 1$, the frequency exceeds ω_{pe} by a factor greater than 1.5. The main conclusion following from Eq. (26) is that the frequency of weakly relativistic waves in electron–positron plasmas is higher than the frequency ω_{pe} of linear oscillations.

3.3.2. Relativistic waves ($\gamma \gg 1$)

It is easily seen that, as in the above-discussed case of waves in a plasma with heavy ions, for relativistic waves in an electron–positron plasma with amplitude much smaller than the limiting amplitude (more exactly, for $\beta\rho \leq 1$), the frequency differs slightly from the frequency of weakly relativistic waves. To find the wave frequency for $\rho \gg 1$, i.e., for $\gamma \gg 1$, we approximately represent the function $V(\psi, \gamma)$ in integral (24) as

$$V(\psi, \gamma) \approx 2\beta\gamma - \sqrt{(\gamma - \psi)^2 - 1} - (\gamma + \psi).$$

We also introduce a new variable $t = (\gamma - \psi) - \sqrt{(\gamma - \psi)^2 - 1}$. Then integral (24) takes the form

$$J(\varepsilon, \gamma) = \int_q^p \frac{(1/t^2 - 1)\,\mathrm{d}t}{2\sqrt{p - t}}\,,$$

where $q = \gamma\,(1 - \beta)$ and $p = \varepsilon + 2q$. This integral is evaluated along the same lines as J_2 (see Sec. 3.2.2). Assuming that $\rho \gg 1$, for the frequency we obtain

$$\omega(\varepsilon, \gamma) \approx \omega_{\mathrm{pe}}\pi\,\sqrt{\gamma\varepsilon}/2. \tag{27}$$

The dependence of the frequency of nonlinear Langmuir waves on the parameters ε and γ in an electron–positron plasma turns out to be the same as for ultrarelativistic waves ($\rho \gg \mu$) in a plasma with heavy ions.

Generalizing the results obtained for weakly relativistic and relativistic waves, we arrive at the conclusion that in electron–positron plasmas, the frequency of nonlinear Langmuir waves is always higher than the frequency of linear oscillations.

4. BASIC CONCLUSIONS

In this paper, we obtained analytical expressions for the frequency of nonlinear Langmuir waves in the entire variation range of the parameters μ, γ, and ε. The main conclusion which follows from the obtained formulas and which should be emphasized first of all is that allowance for the motion of ions is of fundamental importance when studying nonlinear Langmuir waves. It is seen that the dependences of the frequency on the amplitude (ε) and the phase velocity (γ) of waves, as well as on the parameter μ characterizing the ion component of the plasma, are essentially different in the following two limiting cases: (i) for a plasma with infinitely heavy, motionless ions ($\mu \to \infty$), the frequency is always lower than the frequency ω_{p0} of linear waves and monotonically decreases with increasing parameter $\rho = \gamma\varepsilon$; (ii) for an electron–positron plasma in which the ions and electron masses are identical ($\mu = 1$) and, hence, the motion of ions (in this case, positrons) proceeds in the same manner as that of electrons, the frequency is always higher than the frequency of linear oscillations and monotonically increases with increasing ρ. For the intermediate case where the plasma parameters are such that $\mu \gg 1$, the dependence of the frequency on the wave characteristics is fairly complicated and is given by Eqs. (19), (21), and (22).

Another very important conclusion is that, in general, the frequency depends on all three parameters of the problem, i.e., γ, ε, and μ. For fixed μ, the frequency is determined by the product $\rho = \gamma\varepsilon$, i.e., depends equally on both the velocity and amplitude of waves.

We also note some interesting facts. First, weakly relativistic nonlinear plasma waves ($\beta \ll 1$) and small-amplitude relativistic waves for which the condition $\rho = \gamma\varepsilon \leq \beta \leq 1$ holds have an oscillation frequency close to that of linear oscillations in the plasma. Second, for an electron–proton plasma, which is the most widespread in nature, the frequency ω of nonlinear waves (as follows from Eqs. (21) and (22)) differs from the electron plasma frequency ω_{p0} by a factor smaller than an order of magnitude (by a maximum of 7 times) if the parameter ρ varies in a rather wide range $0 \leq \rho \leq 10^5$. Only outside this interval, for $\rho > 10^5$, the wave frequency becomes higher than ω_{p0} and monotonically increases as $\rho^{1/2}$ with increasing ρ.

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